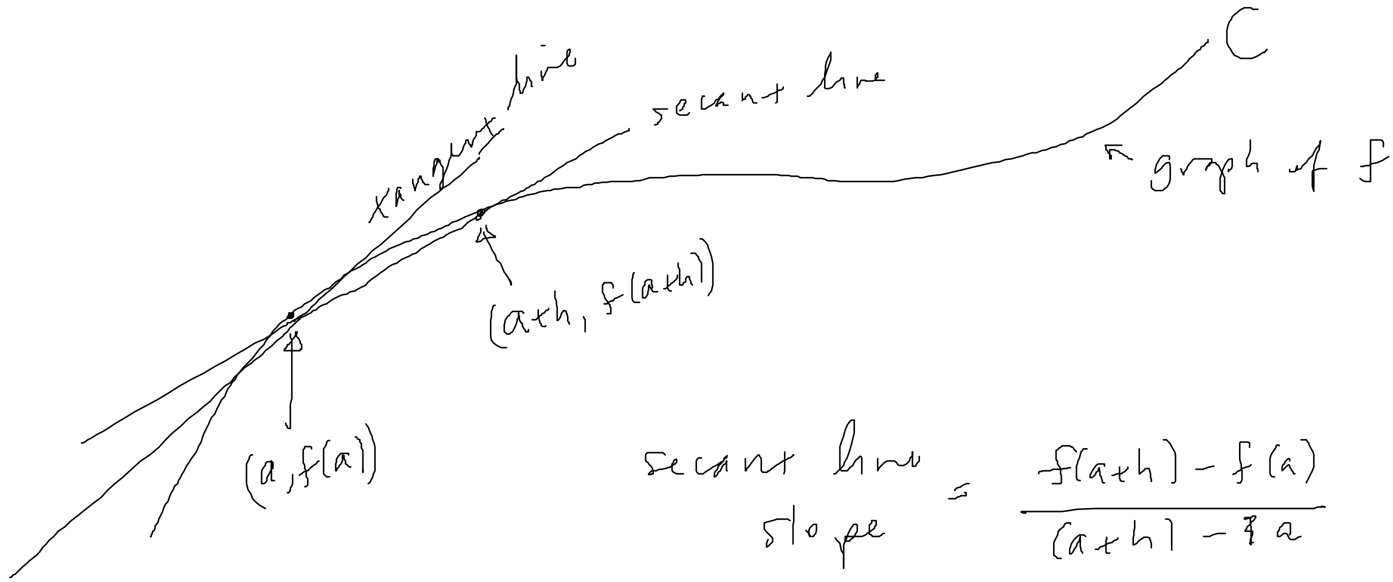
Mat 135, Oct 4 2004

§2.7 Tangents, Velocities, and other rates of change

Recall from §2.1 that we approximated the tangent line to a curve by secant lines



$$\text{secant line slope} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

$\Rightarrow$ secant line: $y = \left(\dfrac{f(a+h) - f(a)}{a+h - a}\right)(x-a) + f(a)$

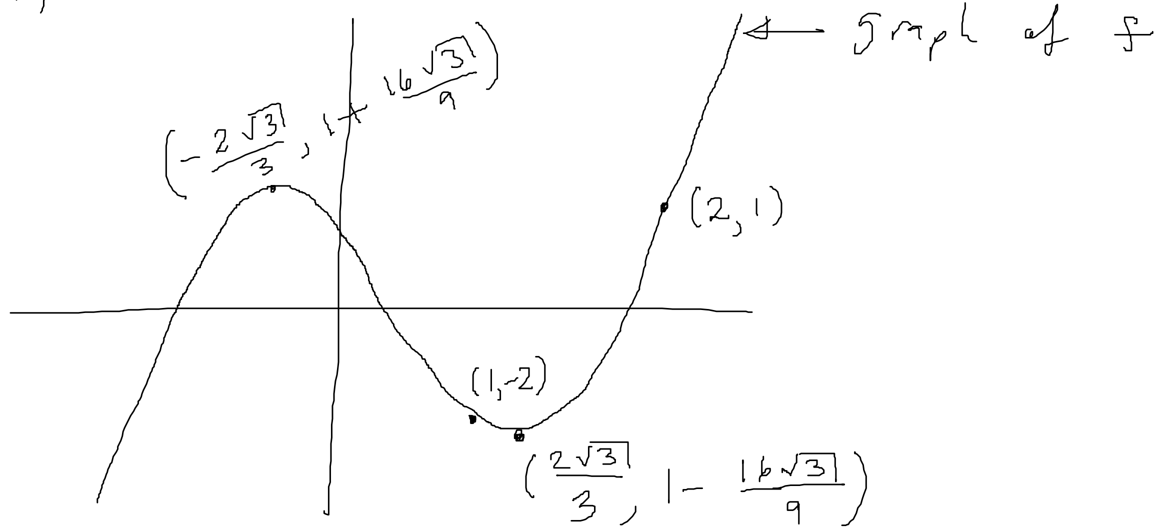
by taking h smaller + smaller, we get a tangent line, if one exists.

tangent line: $y = \left[\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{a+h - a}\right](x-a) + f(a)$

Ex: Consider the graph of

$$f(x) = x^3 - 4x + 1$$

Find the tangent line at $(1,-2)$ and at $(2,1)$



Since I'm asked for two tangent lines, I'll do the calculation w/o fixing a first. Afterwards, I'll set $a=1$ to get one line and $a=2$ to get the other.

secant line slope $= \dfrac{f(a+h)-f(a)}{a+h-a}$

$$= \frac{\left[(a+h)^3 - 4(a+h) + 1\right] - \left[a^3 - 4a + 1\right]}{h}$$

$$= \frac{\left[a^3 + 3a^2h + 3ah^2 + h^3 - 4a - 4h + 1\right] - \left[a^3 - 4a + 1\right]}{h}$$

$$= \frac{3a^2h + 3ah^2 + h^3 - 4h}{h}$$

note! the dangerous h in the denom. cancelled out!

$$= 3a^2 + 3ah + h^2 - 4$$

tangent line: $y = \left[\lim_{h\to 0} \dfrac{f(a+h)-f(a)}{a+h-a}\right](x-a) + f(a)$

$$y = \left[\lim_{h\to 0} 3a^2 + 3ah + h^2 - 4\right](x-a) + f(a)$$

$$y = \left[3a^2 - 4\right](x-a) + f(a)$$

Now $a = 1$, $f(a) = -2$

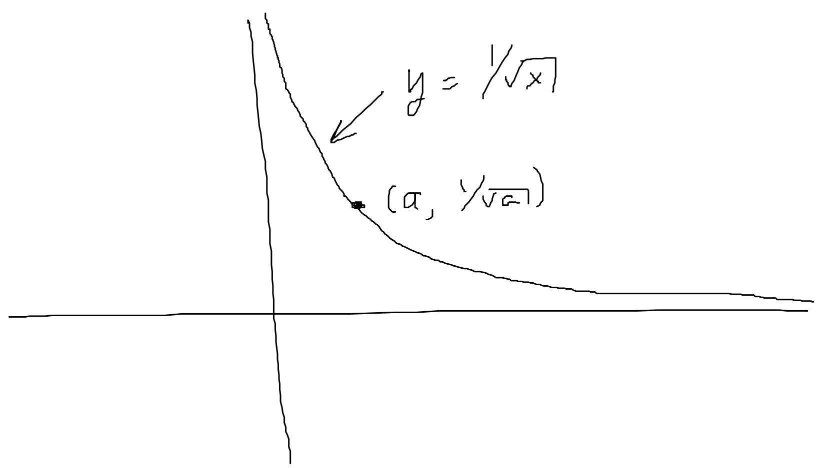
tangent line: $y = (3\cdot(1)^2 - 4)(x-1) - 2$

$$\boxed{y = -1(x-1) - 2}$$

$a = 2$, $f(a) = 1 \Rightarrow$ tangent line $\boxed{y = 8(x-2) + 1}$

ex: a) Find the slope of the tangent to the curve $y = \frac{1}{\sqrt{x}}$ at the point when $x = a$

b) Find equations of the tangent lines at the points $(1, 1)$ and $(4, 1/2)$



$$\text{slope of tangent at } x=a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{a+h-a}$$

$$= \lim_{h \to 0} \frac{1/\sqrt{a+h} - 1/\sqrt{a}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\sqrt{a+h}} - \frac{1}{\sqrt{a}} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sqrt{a} - \sqrt{a+h}}{\sqrt{a}\,\sqrt{a+h}} \right]$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{\sqrt{a} - \sqrt{a+h}}{\sqrt{a}\sqrt{a+h}} \right] \left(\frac{\sqrt{a} + \sqrt{a+h}}{\sqrt{a} + \sqrt{a+h}} \right)$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{a - (a+h)}{\sqrt{a}\sqrt{a+h}} \frac{1}{\sqrt{a} + \sqrt{a+h}}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{-h}{\sqrt{a}\sqrt{a+h}} \frac{1}{\sqrt{a} + \sqrt{a+h}}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{a}\sqrt{a+h}} \frac{1}{\sqrt{a} + \sqrt{a+h}}$$

note! the dangerous h in the denom. cancelled out! ☺

$$= \frac{-1}{a} \frac{1}{2\sqrt{a}} = \frac{-1}{2a\sqrt{a}}$$

$=$

b) tangent line to $y = 1/\sqrt{x}$ at $(1,1)$?

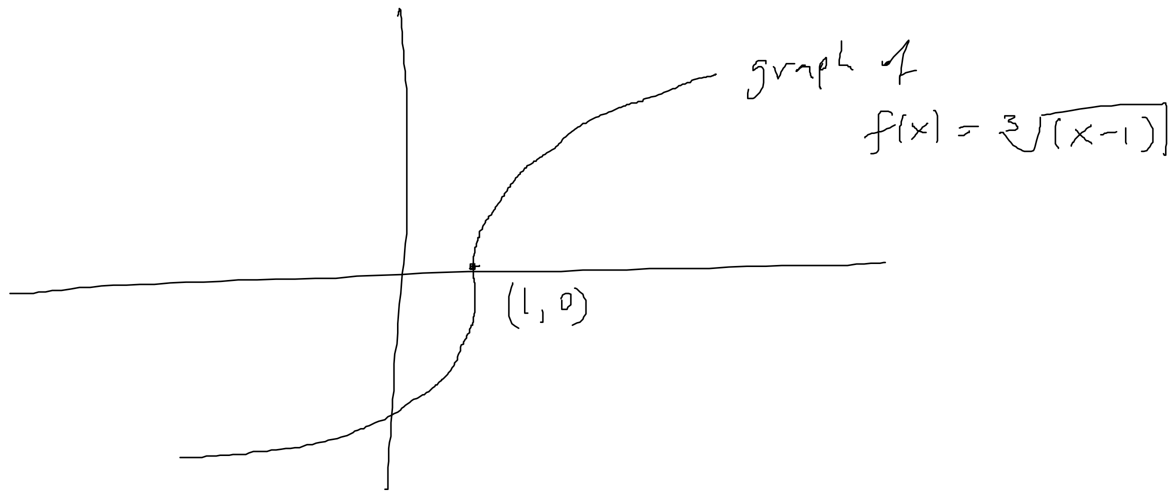
$$y = \frac{-1}{2(1)\sqrt{1}}(x-1) + 1 \Rightarrow \boxed{y = \frac{-1}{2}(x-1) + 1}$$

tangent line at $(4, 1/2)$?

$$y = \frac{-1}{2(4)\sqrt{4}}(x-4) + \frac{1}{2} \Rightarrow \boxed{y = \frac{-1}{16}(x-4) + 1/2}$$

2 Interesting cases

find the tangent line to $\sqrt[3]{x-1}$ at $(1,0)$



tangent line:

$$y = \left[\lim_{h\to 0} \frac{f(1+h) - f(1)}{1+h-1}\right](x-1) + 0$$

$$y = \left[\lim_{h\to 0} \frac{\sqrt[3]{(1+h)-1} - \sqrt[3]{1-1}}{h}\right](x-1)$$

$$y = \left[\lim_{h\to 0} \frac{\sqrt[3]{h} - 0}{h}\right](x-1)$$

$$y = \underbrace{\left[\lim_{h\to 0} \frac{1}{h^{2/3}}\right]}_{=\infty}(x-1)$$

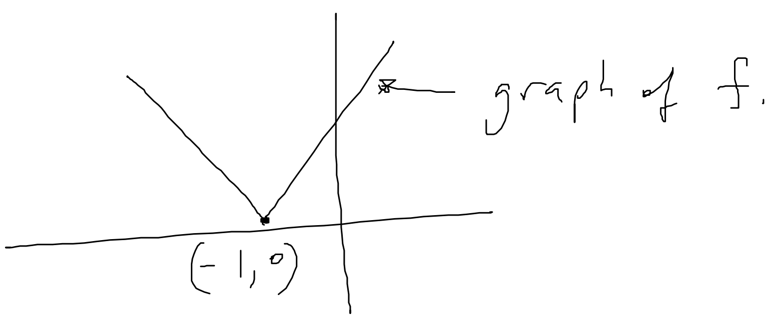
the secant line slope goes to ∞ as $h \to 0$ ☹

can we say that the tangent line at (1, 0) is $y = \infty(x-1)$? no!

In this case, since the slope has an infinite limit, we have a vertical tangent line

tangent line at (1, 0) : $\boxed{x = 1}$

ex: find the tangent line to $|x+1|$ at $(-1, 0)$



first step: find the limit of the secant line slope

$$\frac{f(-1+h)-f(-1)}{1+h-1} = \lim_{h\to 0} \frac{|(-1+h)+1| - |-1+1|}{h}$$

$$= \lim_{h\to 0} \frac{|h|}{h}$$

$$\lim_{h\to 0+} \frac{|h|}{h} = \lim_{h\to 0+} \frac{h}{h} = 1$$

$$\lim_{h\to 0-} \frac{|h|}{h} = \lim_{h\to 0-} \frac{-h}{h} = -1$$

So the secant line slopes tend to 1 if h decreases to 0 and tend to −1 if h increases to 0. This makes sense from the graph of f, of course.

Since $\lim_{h\to 0} \frac{f(-1+h) - f(1)}{(-1+h) - h}$ doesn't exist,

the graph of $f(x) = |1+x|$ does not have a tangent line at $(-1, 0)$.