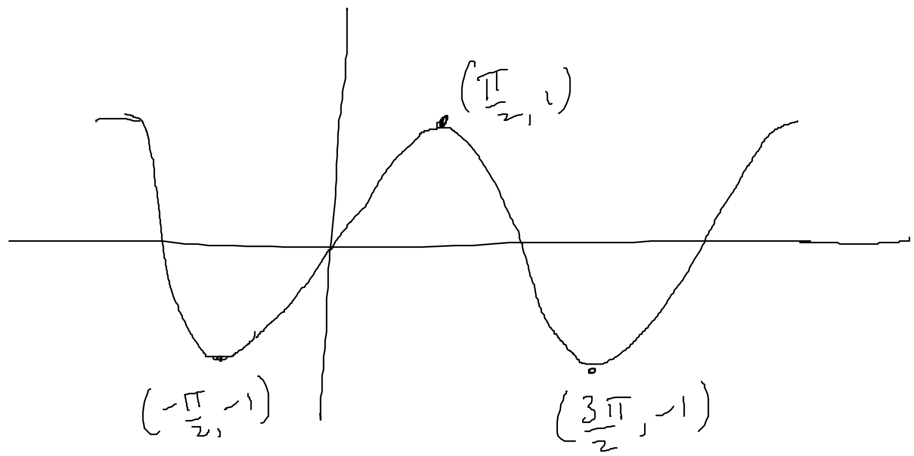
Mat 135, Oct 29 2004

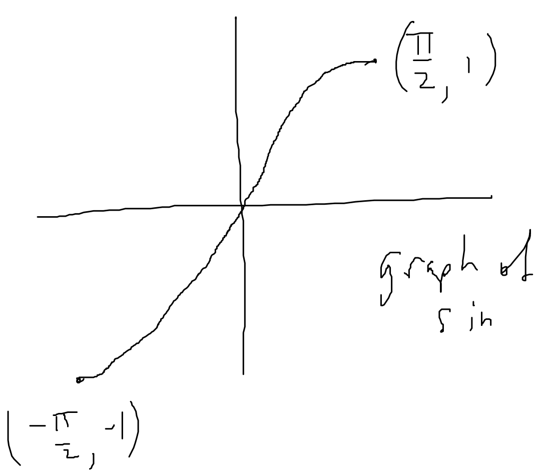
Inverse trig functions & their derivatives

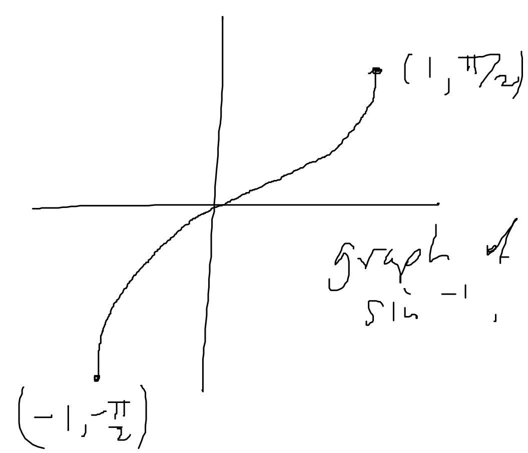


$\sin : (-\infty, \infty) \to [-1, 1]$ is not invertible because each value in $[-1, 1]$ is achieved by infinitely many points in $(-\infty, \infty)$.

$\sin$ is invertible if we restrict the domain

$$\sin : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}] \to [-1, 1]$$





$$\sin^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$$

if $x_0 \in [-\pi/2, \pi/2]$

then $\sin(x_0) = y_0 \iff \sin^{-1}(y_0) = x_0$.

Also, this can be seen as

$\sin^{-1} \circ \sin : [-\pi/2, \pi/2] \to [-\pi/2, \pi/2]$

with $\sin^{-1}(\sin(x)) = x$

or

$\sin \circ \sin^{-1} : [-1, 1] \to [-1, 1]$

with $\sin(\sin^{-1}(x)) = x$.

From the graph of $\sin^{-1}$, it's clear that

$\frac{d}{dx} \sin^{-1}(x)$ is undefined at $x = \pi/2, -\pi/2$

the secant line slopes tend to $+\infty$

if you look at

$$\underbrace{\lim_{h \to 0^-} \frac{\sin^{-1}(\pi/2 + h) - \sin^{-1}(\pi/2)}{h}}$$

these secant line slopes tend to ∞ as $h \to 0-$

Similarly, as $h \to 0+$ $\frac{\sin^{-1}(-\pi/2 + h) - \sin^{-1}(-\pi/2)}{h} \to \infty$

Use implicit differentiation to compute

$\frac{d}{dx} \sin^{-1}(x)$.

$y(x) = \sin^{-1}(x) \iff \sin(y(x)) = x$ and
$y(x) \in [-\pi/2, \pi/2]$

we want to find $\frac{dy}{dx}$ since that will

give us $\frac{d}{dx} \sin^{-1}(x)$. ($y(x) = \sin^{-1}(x)$ after all)

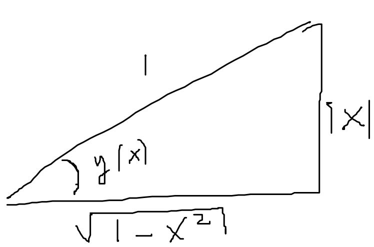
Implicitly differentiate

$\sin(y(x)) = x$ to get

$$\cos(y(x)) \frac{dy}{dx} = 1$$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y(x))}$ we want to

express the denominator in terms of x, not
with $y(x)$.



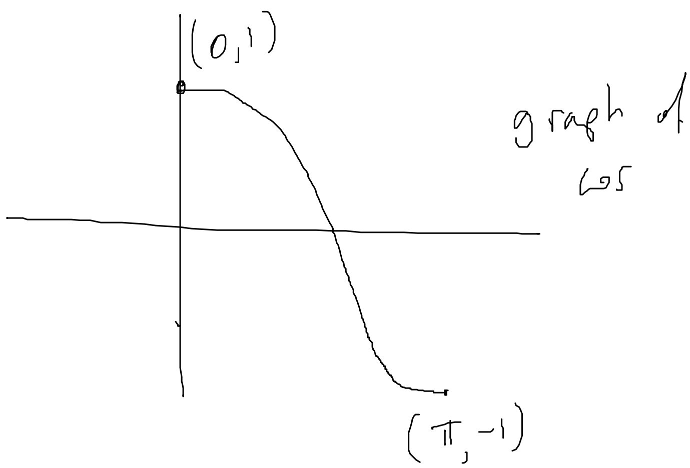
$\sin(y(x)) = x$
$\Rightarrow$ opp $= |x|$ & hyp $= 1$
$\Rightarrow$ adj $= \sqrt{1-x^2} \Rightarrow$
$\cos(y(x)) = \frac{\sqrt{1-x^2}}{1}$

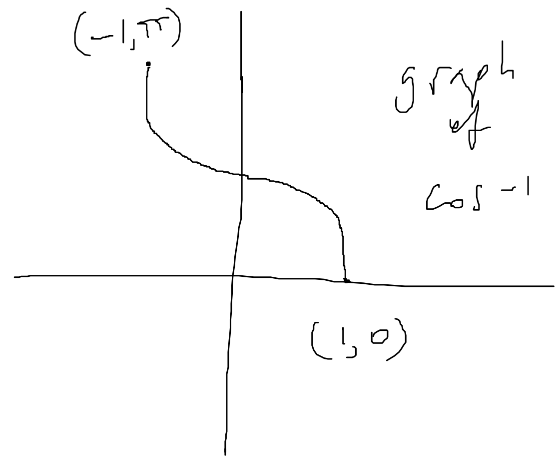
So $$\frac{dy}{dx} = \frac{1}{\cos(y(x))} = \frac{1}{\sqrt{1-x^2}}$$

NOTE: We used $y(x)$ in $[-\pi/2, \pi/2]$ when we wrote

$$\cos(y(x)) = \sqrt{1-x^2}.$$

If $y(x)$ had been in a different interval, then $\cos(y(x))$ might have been negative.





$\cos: (-\infty, \infty) \to [-1, 1]$ is not invertible.

but $\cos: [0, \pi] \to [-1, 1]$ is invertible.

$$\cos^{-1}: [-1, 1] \to [0, \pi]$$

From the graph, $\cos^{-1}$ is not differentiable at ± 1. (Secant line slopes $\to -\infty$.)

Let $y(x) = \cos^{-1}(x)$.

$y(x) = \cos^{-1}(x) \iff \cos(y(x)) = x$ and $y(x) \in [0, \pi]$

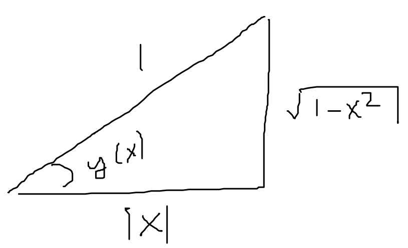
To find $\frac{d}{dx} y(x) = \frac{d}{dx} \cos^{-1}(x)$ we use implicit differentiation on

$$\cos(y(x)) = x.$$

$$\frac{d}{dx} \cos(y(x)) = -\sin(y(x)) \frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{\sin(y(x))}.$$

So we need to figure out $\sin(y(x))$.



$y(x) = \cos^{-1}(x)$

$\iff \cos(y(x)) = x$

$\Rightarrow$ adj $= |x|$ & hyp $= 1$

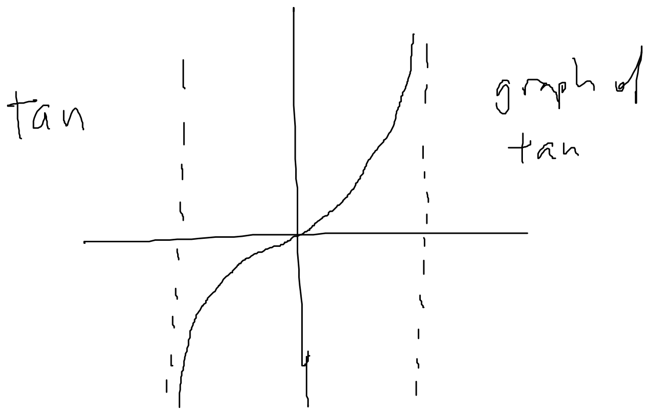
$\Rightarrow$ opp $= \sqrt{1-x^2}$

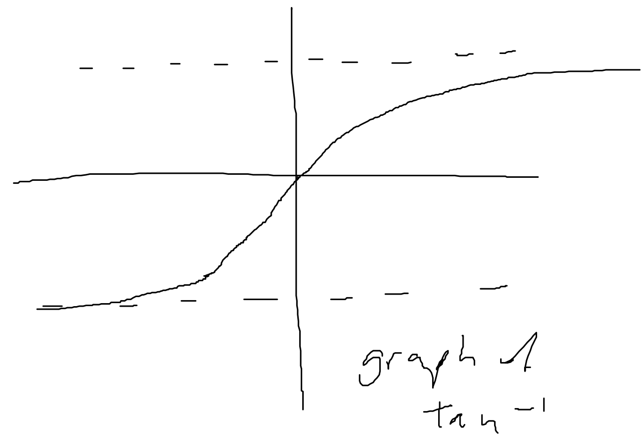
$$\Rightarrow \sin(y(x)) = \frac{\text{opp}}{\text{hyp}} = \sqrt{1-x^2}$$

Note!! I used $y(x)$ in $[0, \pi]$ here since that's what tells me the sign of $\sin(y(x))$.

$$\Rightarrow \frac{d}{dx} \sin^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$$

reality check: $\frac{d}{dx} \sin^{-1}(x) < 0$ as expected.





tan is invertible if restricted to $(-\pi/2, \pi/2)$

$\tan : (-\pi/2, \pi/2) \to (-\infty, \infty)$

then $\tan^{-1} : (-\infty, \infty) \to (-\pi/2, \pi/2)$

expect from graph:

$\frac{d}{dx} \tan^{-1}(x) > 0$ and $\frac{d}{dx} \tan^{-1}(x) \to 0$ as $x \to \infty$

also as $x \to -\infty$.

$y(x) = \tan^{-1}(x) \iff \tan(y(x)) = x$ and $-\frac{\pi}{2} < y(x) < \frac{\pi}{2}$

$$\frac{d}{dx} \tan(y(x)) = (\sec(y(x)))^2 \frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

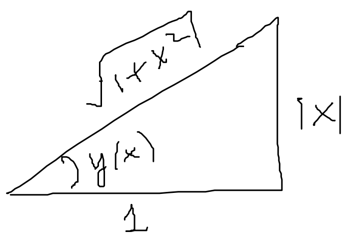
so $$\frac{dy}{dx} = \frac{1}{(\sec(y(x)))^2} = (\cos(y(x)))^2$$

need to find $\cos(y(x))$ ...

we know

$$y(x) = \tan^{-1}(x) \iff \tan(y(x)) = x \quad \text{and} \quad -\frac{\pi}{2} < y(x) < \pi/2$$

Since $-\pi/2 < y(x) < \pi/2$, we know $\cos(y(x)) > 0$



$\tan(y(x)) = x \Rightarrow \text{opp} = |x|$ \$ $\text{adj} = 1$

$\Rightarrow \text{hyp} = \sqrt{1+x^2}$

$\Rightarrow \cos(y(x)) = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{1+x^2}}$

(Note: used $\cos(y(x)) > 0$ here $\nearrow$).

$$\therefore \frac{d}{dx} y(x) = \frac{d}{dx} \tan^{-1}(x) = \left(\frac{1}{\sqrt{1+x^2}}\right)^2 = \frac{1}{1+x^2}$$

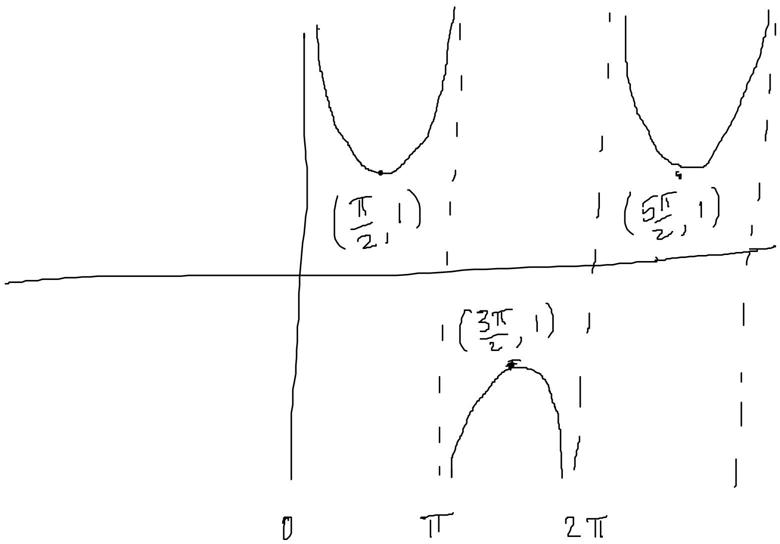
reality check: $\frac{d}{dx} \tan^{-1}(x) > 0$ on $(-\infty, \infty)$ ✓

and tends to 0 as $x \to \infty$

and as $x \to -\infty$.

$\frac{d}{dx} \csc^{-1}(x)$?

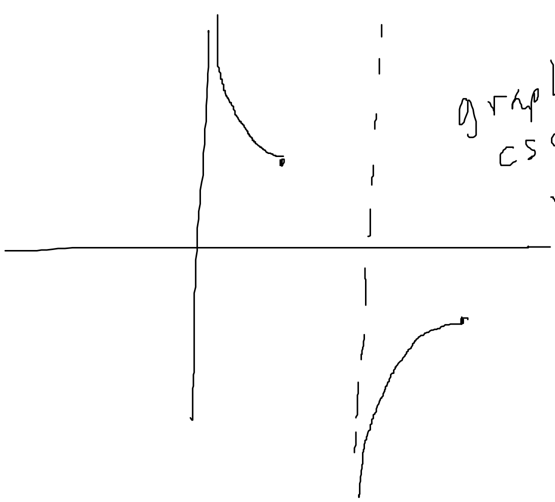
$\csc(x) = \frac{1}{\sin(x)}$



csc is not invertible. To invert it, we need to restrict its domain.

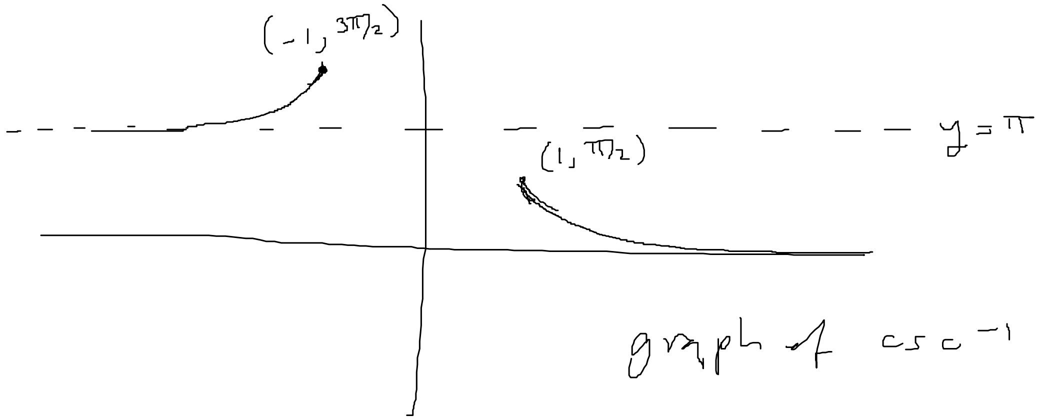
$\csc : (0, \pi/2] \cup (\pi, \frac{3\pi}{2}] \to (-\infty, -1] \cup [1, \infty)$

is invertible.



graph of csc when restricted to $(0, \pi/2] \cup (\pi, 3\pi/2]$

$\csc^{-1} : (-\infty,-1] \cup [1,\infty) \to (0,\pi/2] \cup (\pi, 3\pi/2]$



graph of $\csc^{-1}$

from the graph, there are vertical tangents at
$x=-1$ and $x=+1$.

$$\frac{d}{dx}\csc^{-1}(x) > 0 \text{ if } x<-1$$

$$\frac{d}{dx}\csc^{-1}(x) < 0 \text{ if } x>1$$

and as $x \to -\infty$ $\frac{d}{dx}\csc^{-1}(x) \to 0$

as $x \to +\infty$ $\frac{d}{dx}\csc^{-1}(x) \to 0$.

Use implicit differentiation to compute
$\frac{d}{dx}\csc^{-1}(x)$.

$$y(x) = \csc^{-1}(x) \iff \csc(y(x)) = x$$

$$\frac{d}{dx}\csc(y(x)) = -\csc(y(x))\cot(y(x))\frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

So $$\frac{d}{dx}y(x) = \frac{-1}{\csc(y(x))\cot(y(x))} = \frac{-\sin(y(x))}{\cot(y(x))}$$

Need to find

$\csc(y(x))$ and $\cot(y(x))$ in terms of x.

<u>case 1</u>: $x \geq 1$. then $y(x) \in (0, \pi/2]$

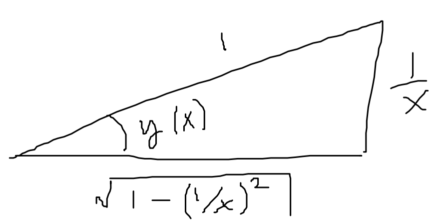
and so $\cos(y(x)) \geq 0$

$\sin(y(x)) \geq 0$

$$y(x) = \csc^{-1}(x) \iff \csc(y(x)) = x$$

$$\iff \frac{1}{\sin(y(x))} = x$$

$$\iff \sin(y(x)) = \frac{1}{x}$$



$\sin(y(x)) = \frac{1}{x}$

$\Rightarrow$ opp $= \frac{1}{x}$ & hyp $= 1$

$\Rightarrow$ adj $= \sqrt{1-(1/x)^2}$

$= \frac{\sqrt{x^2-1}}{|x|} = \frac{\sqrt{x^2-1}}{x}$

So $\cos(y(x)) = \frac{x}{\sqrt{x^2-1}}$

(Note: used $\cos(y(x))$ & $\sin(y(x)) \geq 0$!)

$$\Rightarrow \frac{d}{dx} \csc^{-1}(x) = \frac{-1}{\csc(y(x))\cot(y(x))} = \frac{-(\sin(y(x)))^2}{\cos(y(x))}$$

$$= \frac{-(1/x)^2}{\frac{\sqrt{x^2-1}}{x}} = -\frac{1}{x\sqrt{x^2-1}}$$

Case 2: $x \leq -1$ then $y(x) \in (\pi, 3\pi/2]$

$\Rightarrow \sin(y(x)) \leq 0$ and $\cos(y(x)) \leq 0$

by the same geometric argument,

$$\sin(y(x)) = \frac{1}{x} \ \& \ \cos(y(x)) = \frac{\sqrt{x^2-1}}{x}$$

(check these are negative!)

and $\frac{d}{dx} \csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$

So we've found that $\frac{d}{dx} \csc^{-1}(x) = \frac{-1}{x\sqrt{x^2-1}}$

whether $x \leq -1$ or $x \geq 1$

reality check: $\frac{d}{dx} \csc^{-1}(x) > 0$ if $x \leq -1$ ✓

< 0 if $x \geq 1$ ✓

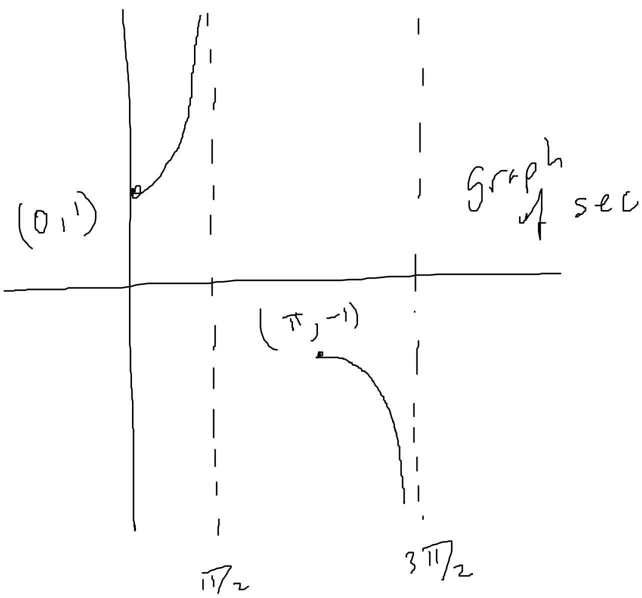
$\to 0$ if $x \to \infty$ ✓

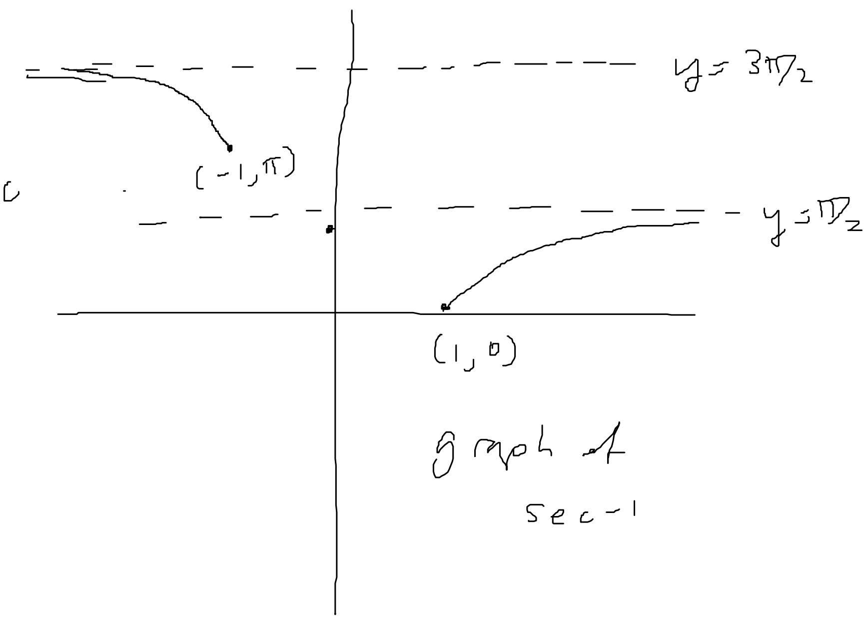
$\to 0$ if $x \to -\infty$ ✓

Similarly, for $\sec^{-1}$

$\sec : [0, \pi/2) \cup [\pi, 3\pi/2) \to (-\infty, -1] \cup [1, \infty)$

$\sec^{-1} : (-\infty, -1] \cup [1, \infty) \to [0, \pi/2) \cup [\pi, 3\pi/2)$





from graph: vertical tangents at $x = \pm 1$

$$\frac{d}{dx}\sec^{-1}(x) < 0 \quad \text{if } x \le -1$$

$$> 0 \quad \text{if } x \ge 1$$

$$\to 0 \quad \text{as } x \to \infty$$

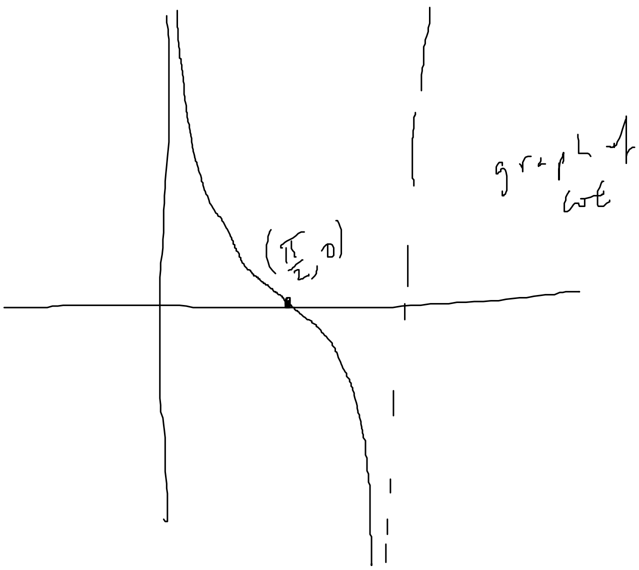
$$\to 0 \quad \text{as } x \to -\infty$$

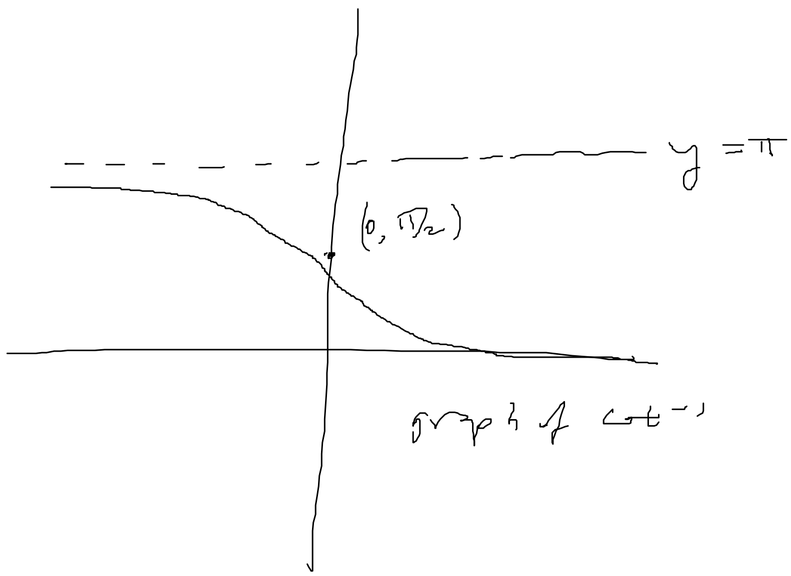
implicit differentiation + trig $\Rightarrow \dfrac{d}{dx}\sec^{-1}(x) = \dfrac{1}{x\sqrt{x^2-1}}$

Similarly for $\cot^{-1}$

$$\cot : (0, \pi) \to (-\infty, \infty)$$

$$\cot^{-1} : (-\infty, \infty) \to (0, \pi)$$





from graph: $\frac{d}{dx} \cot^{-1}(x) < 0$ on $(-\infty, \infty)$

$\to 0$ as $x \to \infty$

$\to 0$ as $x \to -\infty$

implicit differentiation + trig $\Rightarrow$ $\frac{d}{dx} \cot^{-1}(x) = \frac{-1}{1+x^2}$

passes reality checks!