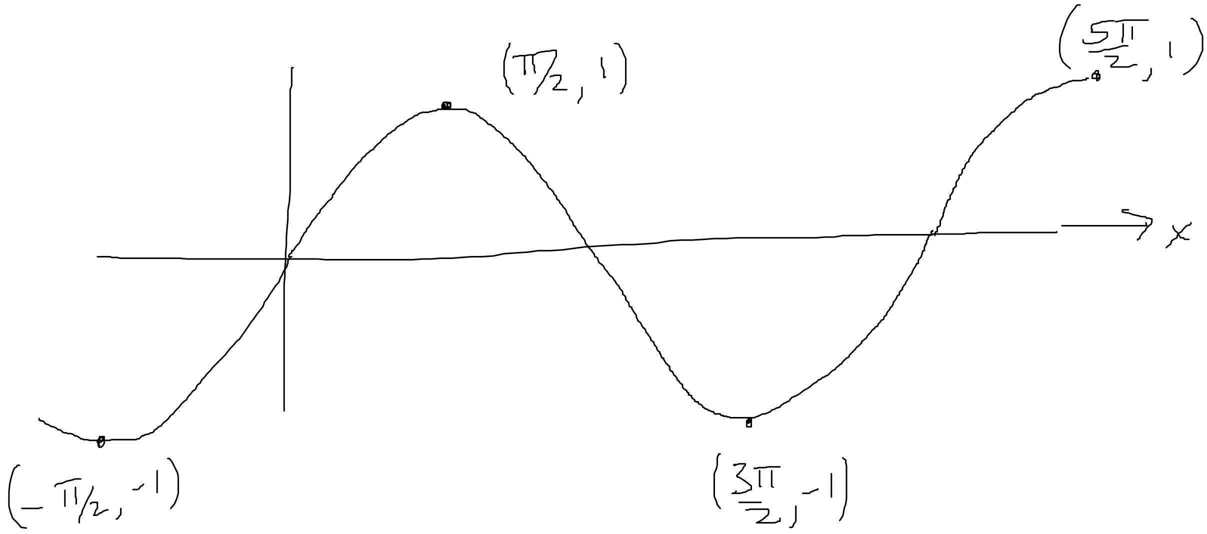
§ 3.4 Differentiation of Trig functions.

$$\frac{d}{dx} \sin(x) = \cos(x)$$

$$\frac{d}{dx} \cos(x) = -\sin(x).$$

The book explains why $\frac{d}{dx}\sin(x) = \cos(x)$ and leaves $\frac{d}{dx}\cos(x) = -\sin(x)$ as an exercise (# 20)

I will try to argue the reasonableness of these facts; why you should not be surprised to hear that $\frac{d}{dx}\sin(x) = \cos(x)$ & $\frac{d}{dx}\cos(x) = -\sin(x)$.



graph of $\sin(x)$.

From the graph of $\sin(x)$ you know:

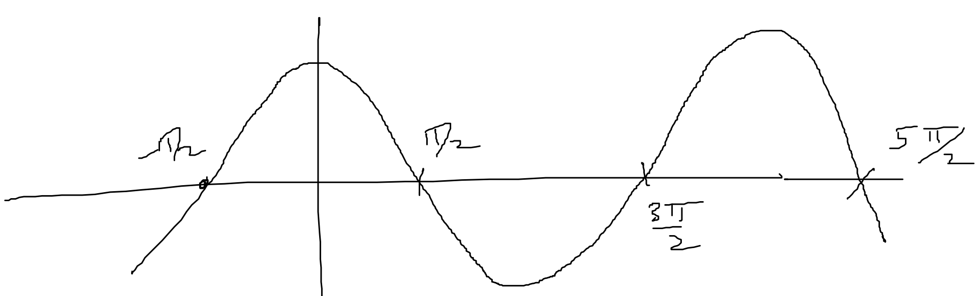
positive tangent line slopes : $-\frac{\pi}{2} < x < \frac{\pi}{2}$, $\frac{3\pi}{2} < x < \frac{5\pi}{2}$

negative tangent line slopes : $\frac{\pi}{2} < x < \frac{3\pi}{2}$

horizontal tangent lines : $x = -\pi/2,\ \pi/2,\ 3\pi/2,\ \frac{5\pi}{2}$

So
$$\frac{d}{dx}\sin(x) \begin{cases} = 0 & \text{at } -\pi/2,\ \pi/2,\ 3\pi/2,\ 5\pi/2 \\ > 0 & \text{on } (-\pi/2, \pi/2) \cup (3\pi/2, 5\pi/2) \\ < 0 & \text{on } (\pi/2, 3\pi/2). \end{cases}$$

certainly $\cos(x)$ fits that description:

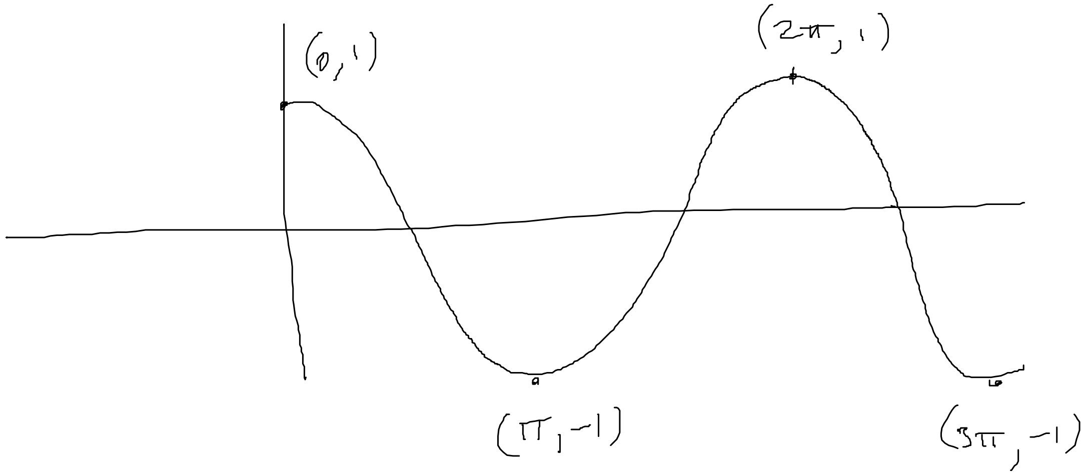


of course, $2\cos(x)$ and $10\cos(x)$ also fit that description.

You need to go through the book's proof to see $\frac{d}{dx}\sin(x) = \cos(x)$

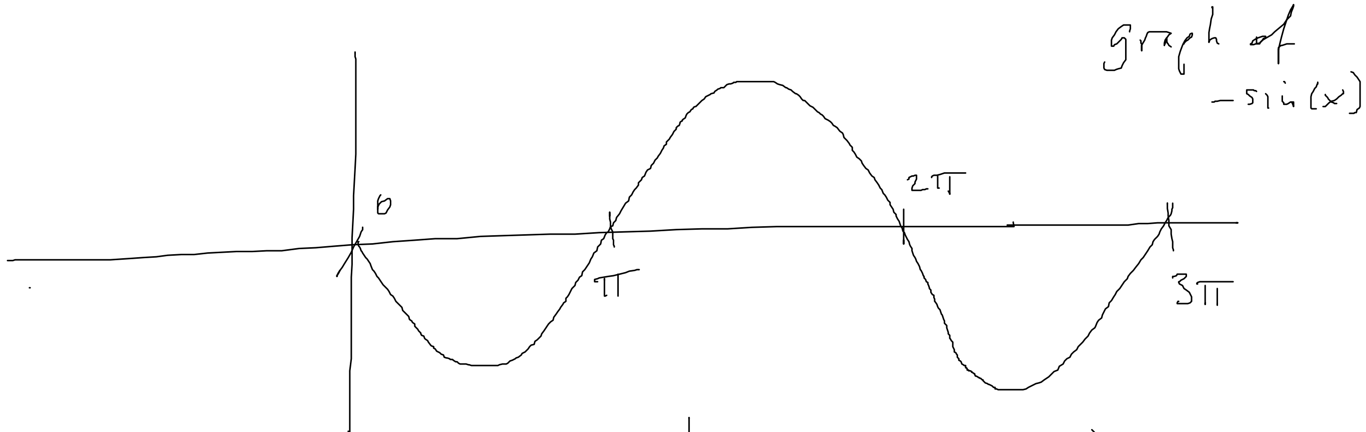
Similarly, we understand

$\frac{d}{dx}\cos(x)$ by looking at the graph of $\cos(x)$:



$$\frac{d}{dx}\cos(x) \begin{cases} = 0 & \text{at } 0, \pi, 2\pi, 3\pi \\ < 0 & \text{on } (0,\pi) \cup (2\pi, 3\pi) \\ > 0 & \text{on } (\pi, 2\pi) \end{cases}$$

certainly $-\sin(x)$ fits that description:



So you're not shocked to learn $\frac{d}{dx}\cos(x) = -\sin(x)$.

You must know

$$\frac{d}{dx}\cos(x) \quad \text{and} \quad \frac{d}{dx}\sin(x) \quad \text{by}$$

heart. You can figure out the derivatives of the other 4 trig functions using the quotient rules,

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)}$$

$$= \frac{\left(\frac{d\sin}{dx}\right)\cos(x) - \left(\frac{d}{dx}\cos x\right)\sin(x)}{(\cos(x))^2}$$

$$= \frac{\cos(x)\,\cos(x) - (-\sin(x))\sin(x)}{(\cos(x))^2}$$

$$= \frac{1}{(\cos(x))^2} = (\sec(x))^2$$

$$\frac{d}{dx}(\cot x) = -(\csc(x))^2$$

$$\frac{d}{dx}(\csc x) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}(\sec x) = \sec(x)\tan(x).$$

find

ex # 10 $\frac{d}{dx} \frac{1+\sin(x)}{x+\cos(x)}$

$$= \frac{\cos(x)\,(x+\cos(x)) - (1+\sin(x))(1-\sin(x))}{(x+\cos(x))^2}$$

$$= \frac{x\cos(x) + (\cos(x))^2 - 1 + (\sin(x))^2}{(x+\cos(x))^2}$$

$$= \frac{x\cos(x)}{(x+\cos(x))^2}$$

ex # 23 find eqn of tangent line at given point.

$y = x + \cos(x) \qquad (0,1)$

$\frac{dy}{dx} = 1 - \sin(x) \Rightarrow \frac{dy}{dx}(0) = 1 - \sin(0) = 1$

tangent line: $y = 1(x-0) + 1$

$$\boxed{y = x + 1}$$

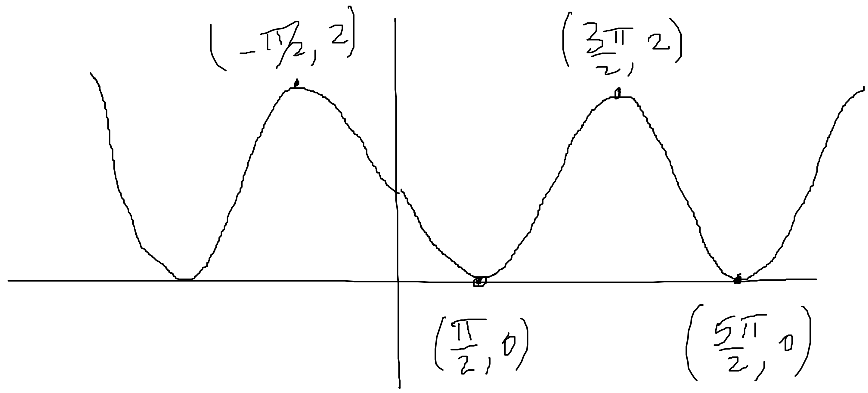
I want to plot $f(x) = x + \cos(x)$.

To do this, I first figure out where the function is increasing and where the function is decreasing. (I know the graph will look like something oscillating around $y=x$, but I want better information than that...)

$f'(x) = 1 - \sin(x)$ $f'(x) = 0$ if $\sin(x) = 1$

that is $x = \frac{\pi}{2} + n \cdot 2\pi$

Note that $f'(x) > 0$ for all other values of x.

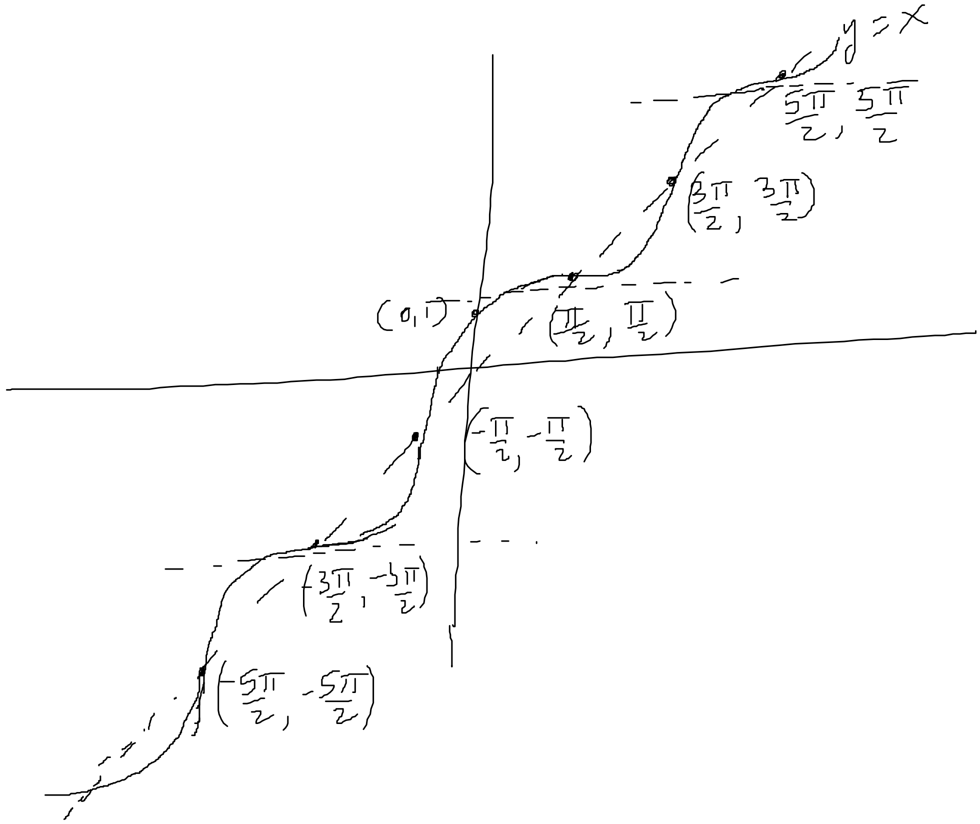


Graph of f'

To help with the graphing, I'll evaluate $f(x)$ at the points where I <u>know</u> there are horiz. tangents

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} + \cos\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$$

in general, $f\left(\frac{\pi}{2} + n2\pi\right) = \frac{\pi}{2} + n2\pi$ for all integers n.



the other crossing points are found by seeking x such that $f(x) = x$.

That is $f(x) = x + \cos(x) = x$

$$\Rightarrow x = 3\frac{\pi}{2} + n \cdot 2\pi \text{ or } x = \frac{\pi}{2} + n \cdot 2\pi$$

n any integer

$f'(x) = 2$ for $x = \frac{3\pi}{2} + n \cdot 2\pi$

(I drew my graph a little too steeply)

Let's plot

$$f(x) = x + \frac{2}{\sqrt{3}} \sin(x)$$

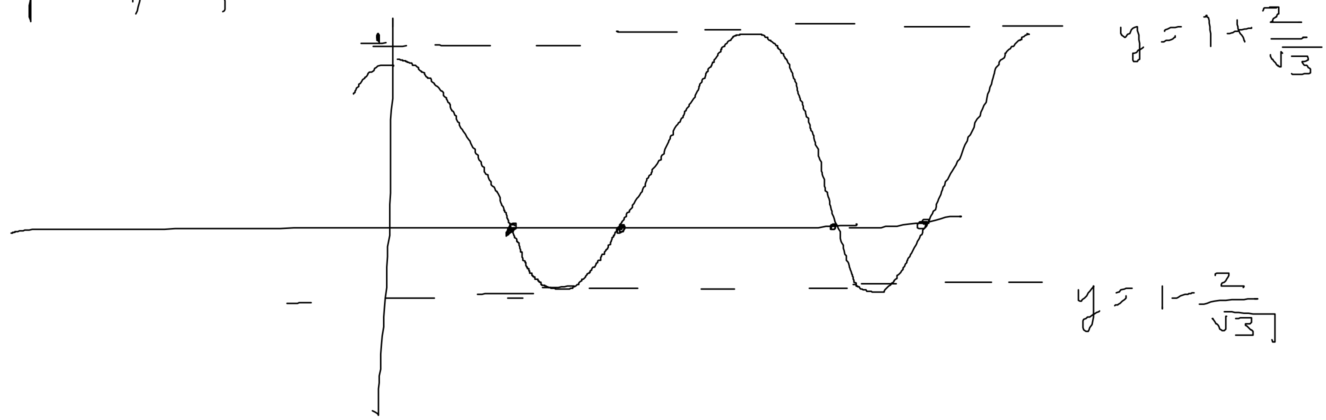
and $f(x) = x + \sin(x)\,\frac{1}{2}$

for practice.

$$f(x) = x + \frac{2}{\sqrt{3}} \sin(x) \Rightarrow f'(x) = 1 + \frac{2}{\sqrt{3}} \cos(x).$$

Since $\frac{2}{\sqrt{3}} > 1$ we see that

the graph of f' is



and there are points where $f'(x) = 0$

$f'(x) = 0$ if $\cos(x) = -\frac{\sqrt{3}}{2}$ that is

$$x = \frac{5\pi}{6} + n \cdot 2\pi \text{ or } x = \frac{7\pi}{6} + n \cdot 2\pi$$

Also, I'll look for points where the graph of $f(x) = x + \frac{2}{\sqrt{3}} \sin(x)$ crosses $y = x$.

That is $x + \frac{2}{\sqrt{3}} \sin(x) = x \Rightarrow x = 0 + n \cdot 2\pi$

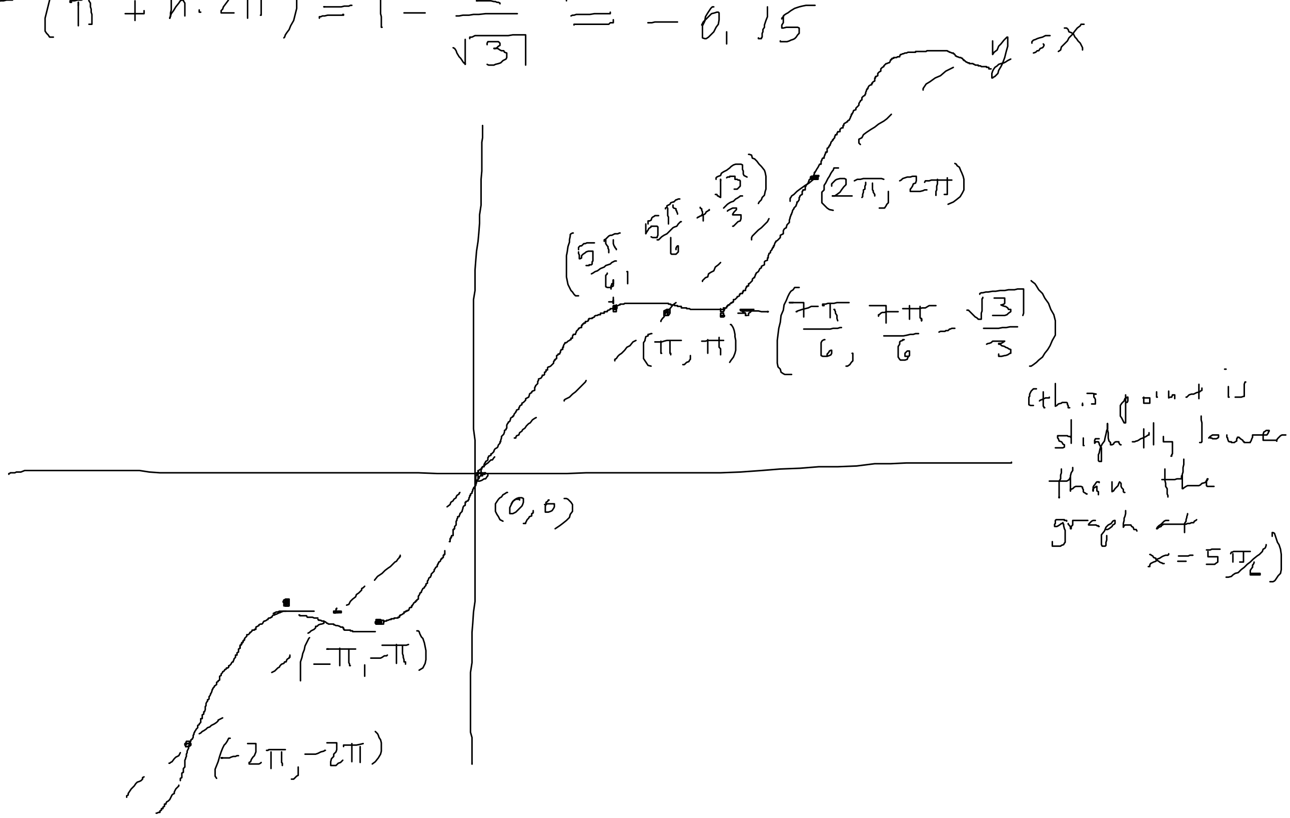
or

$x = \pi + n \cdot 2\pi$

I evaluate $f'(x)$ at these crossing points.

$$f'(0 + n \cdot 2\pi) = 1 + \frac{2}{\sqrt{3}} \cong 2.15$$

$$f'(\pi + n \cdot 2\pi) = 1 - \frac{2}{\sqrt{3}} \cong -0.15$$



Finally, let's graph $f(x) = x + \frac{1}{2}\sin(x)$.

Q1. Horizontal tangents?

$f'(x) = 1 + \frac{1}{2}\cos(x)$ never equals zero.

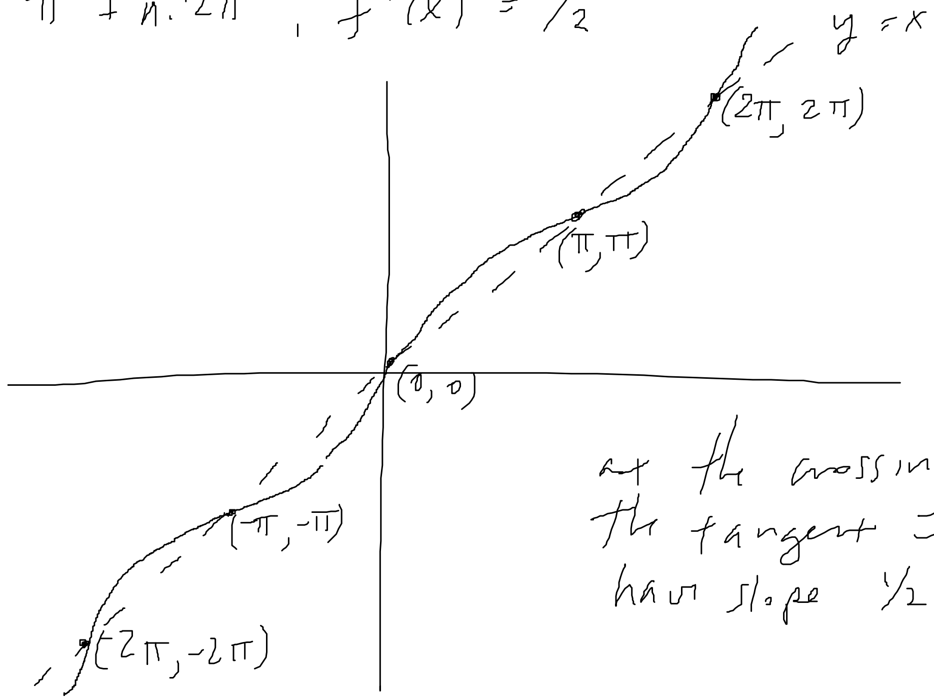
$f'(x) > 0$ for all x

Q2: Crossing $y = x$? Occurs when $\sin(x) = 0$

That is $x = 0 + n \cdot 2\pi$ or $x = \pi + n \cdot 2\pi$

at $0 + n \cdot 2\pi$, $f'(x) = 3/2$

at $\pi + n \cdot 2\pi$, $f'(x) = 1/2$



at the crossing points the tangent lines have slope $1/2$ or $3/2$

#30 Find the points on the curve $y = \dfrac{\cos(x)}{2+\sin(x)}$

at which the tangent is horizontal.

$$\frac{d}{dx}\ \frac{\cos(x)}{2+\sin(x)} = -\ \frac{2\sin(x)+1}{(2+\sin(x))^2}$$

equals zero whenever $2\sin(x) + 1 = 0$ i.e.

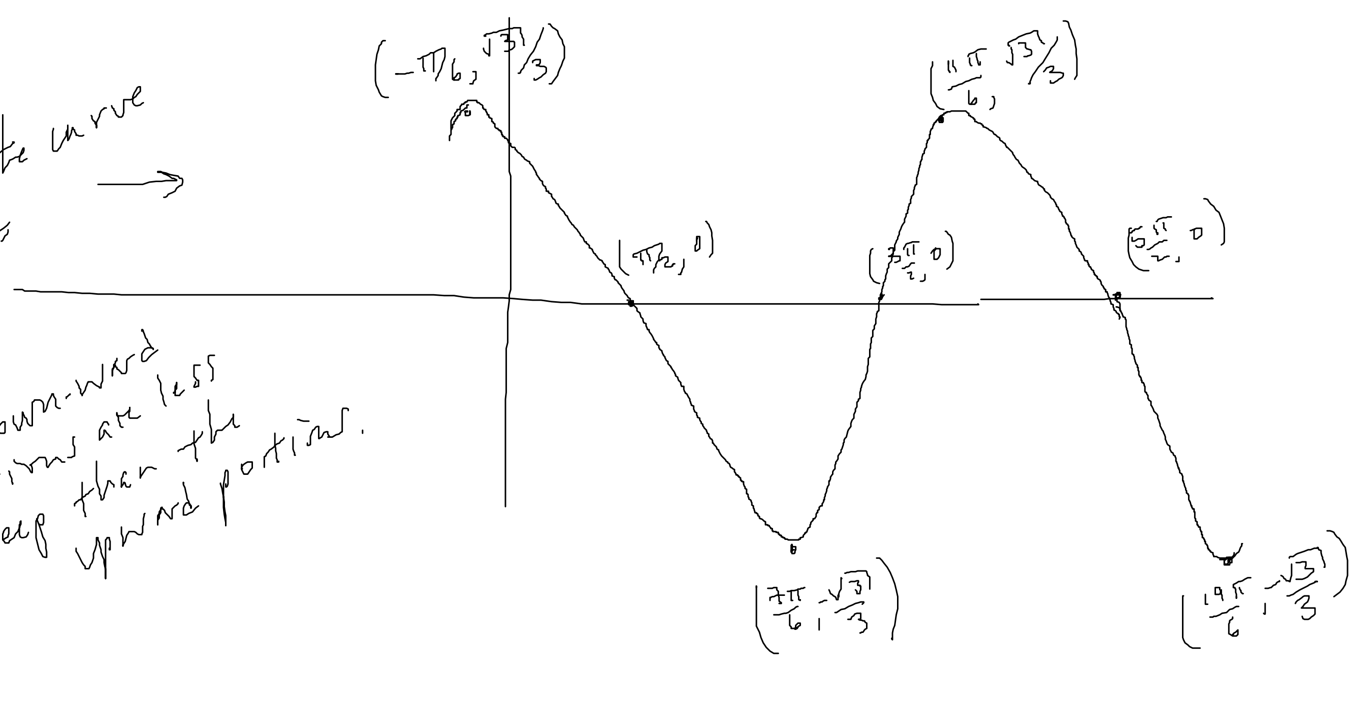
at $x = -\pi/6 + n \cdot 2\pi$ and $-\frac{5\pi}{6} + n \cdot 2\pi$

where n is any integer.

$$\frac{\cos(-\pi/6)}{2+\sin(-\pi/6)} = \frac{\sqrt{3}}{2} \qquad \frac{\cos(-5\pi/6)}{2+\sin(-5\pi/6)} = -\frac{\sqrt{3}}{3} \cong -.58$$

the curve $\longrightarrow$

$y =$



the down-ward portions are less steep than the upward portions.

In proving that

$$\frac{d}{dx}\sin(x) = \cos(x) \quad \text{and} \quad \frac{d}{dx}\cos(x) = -\sin(x)$$

One needs to know

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0}\frac{\cos(x)-1}{x} = 0$$

These limits may pop up in problems, so it's good to see them now.

ex: $$\lim_{x\to 0}\frac{\sin(3x)}{x} = \lim_{x\to 0} 3\,\frac{\sin(3x)}{3x}$$

let $y = 3x$ then $x \to 0 \Rightarrow y \to 0$

$$= \lim_{y\to 0} 3\,\frac{\sin(y)}{y} = 3$$

ex: $$\lim_{\theta\to 0}\frac{\sin(\cos(\theta))}{\sec(\theta)} = \lim_{\theta\to 0}\frac{\sin(\cos(\theta))}{1/\cos(\theta)}$$

$$= \lim_{\theta\to 0}\cos(\theta)\sin(\cos(\theta))$$

as $\theta \to 0$ $\cos(\theta) \to 1$ let $y = \cos(\theta)$

$$= \lim_{y\to 1} y\sin(y) = \sin(1)$$

#42 $$\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta + \tan(\theta)} = \lim_{\theta\to 0} \frac{\sin(\theta)}{\theta + \frac{\sin(\theta)}{\cos(\theta)}}$$

We know $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta} = 1$, so I'm going to multiply by $\frac{1/\theta}{1/\theta}$

$$= \lim_{\theta\to 0} \frac{\frac{1}{\theta}}{\frac{1}{\theta}} \frac{\sin\theta}{\theta + \frac{\sin\theta}{\cos\theta}}$$

$$= \lim_{\theta\to 0} \frac{\frac{\sin\theta}{\theta}}{\frac{1}{\theta}\left(\theta + \frac{\sin\theta}{\cos\theta}\right)}$$

$$= \lim_{\theta\to 0} \frac{\frac{\sin\theta}{\theta}}{1 + \frac{\sin\theta}{\theta}\frac{1}{\cos\theta}}$$

$$= \frac{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}{1 + \lim_{\theta\to 0}\frac{\sin\theta}{\theta}\lim_{\theta\to 0}\frac{1}{\cos\theta}}$$ by the limit rules

$$= \frac{1}{1 + 1\cdot 1} = \frac{1}{2}$$