Mat 135, Oct 20 2004

§3.3 Rates of Change in the Natural & Social Sciences

#11 If a ball is thrown vertically upward with a velocity of 80 ft/sec then its height after t seconds is

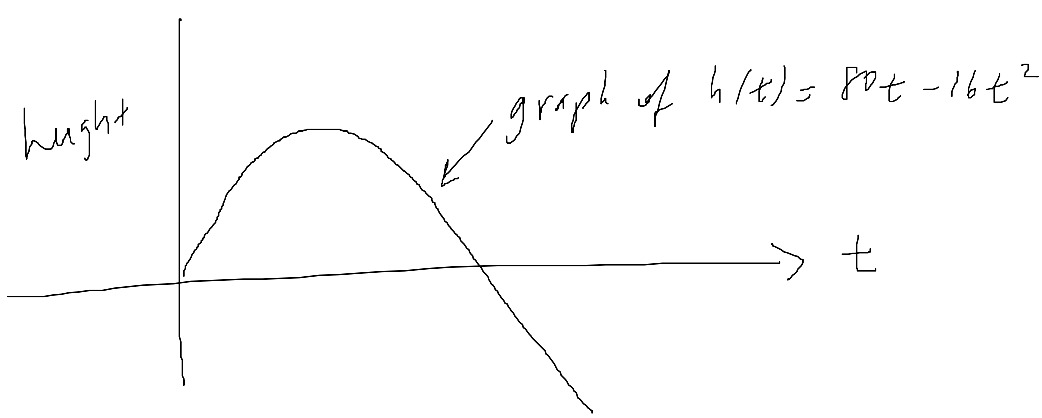
$$h(t) = 80t - 16t^2$$

Q: What is the maximum height reached by the ball?

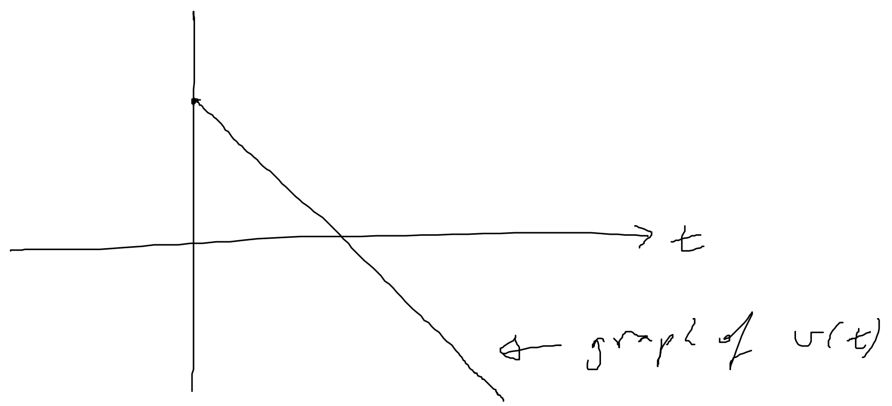
Q: When does it fly upwards? Downwards?

Q: When is the ball at the same height as when it started? What is its speed then?

Q. What is the velocity of the ball when it is 96 feet above the ground? on its way up? on its way down?



velocity $v(t) = \frac{dh}{dt} = 80 - 32t$



Initial velocity $= v(0) = 80$ ✓ as expected

$v(t) > 0$ for $t < \frac{80}{36} = \frac{5}{2}$

Q1: maximum height occurs at $t = 5/2$

$h(5/2) = 80\left(\frac{5}{2}\right) - 16\left(\frac{5}{2}\right)^2 = 100$ feet.

Q2: travelling up for $t < 5/2$, travelling down for $t > 5/2$

Q3: ball at same height as beginning when

$h(t_0) = h(0)$ i.e. $80t_0 - 16t_0^2 = 0$

$\Rightarrow t_0 = 5$ seconds

Speed at 5 seconds?

$v(5) = 80 - 32 \cdot 5 = -80$

coming down w/ same speed! (Mental note: don't shoot guns in the air as celebration)

Q4: What is velocity of ball when it reaches 96 feet? First, when does it reach 96 feet?

$$h(t_0) = 96 \ldots$$

$$80 t_0 - 16 t_0^2 = 96$$

$$\Rightarrow t_0 = 2 \text{ or } 3$$

going up at time 2, going down at 3 seconds.

$v(2) = 16$

$v(3) = -16$

Note: $v(2) < v(0)$ as expected since the ball is slowing down as it rises.

#25 One molecule of product C is formed from one molecule of the reactant A and one molecule of the reactant B.

$$A + B \longrightarrow C$$

If A and B have equal concentrations initially $[A] = [B] = a$ moles/L then

$$[C] = \frac{a^2 kt}{akt + 1}$$

where k is a constant

Q: what is the initial concentration of C?

Q: as $t \to \infty$ what does C's concentration tend to?

Q: find the rate of reaction at time t

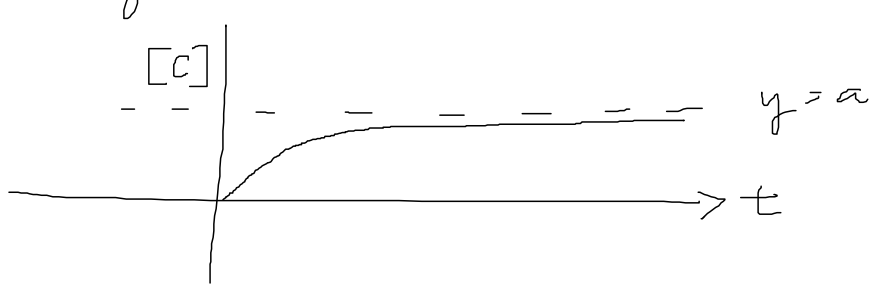
Q: as $t \to \infty$ what does the rate of reaction tend to?

Q1: at time 0, $[C] = \dfrac{a^2k \cdot 0}{ak \cdot 0 + 1} = 0$

initially there's no product C.

Q2: as $t \to \infty$ $\displaystyle\lim_{t \to \infty} [C] = \lim_{t \to \infty} \frac{a^2kt}{akt+1} = \frac{a^2k}{ak} = a$

so the concentration of C tends to a as $t \to \infty$. This makes sense since the initial concentration of A & B is a moles/litre and you need one A and one B to make one C. So all A & B will eventually get used up, creating a moles/litre of C.



Q3:

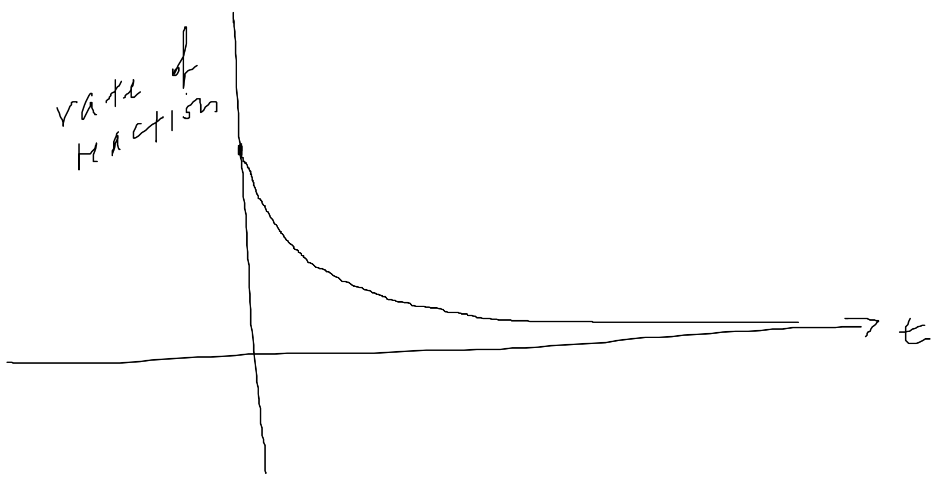
rate of reaction? that's chemistry-speak for $\frac{d}{dt}[C]$.

rate of reaction $= \frac{d}{dt} \frac{a^2kt}{akt+1}$ Quotient rule!

$$= \frac{a^2k(akt+1) - ak(a^2kt)}{(akt+1)^2}$$

$$= \frac{a^3k^2t + a^2k - a^3k^2t}{(akt+1)^2}$$

$$= \frac{a^2k}{(akt+1)^2}$$



The reaction rate is always >0. As $t \to \infty$ the reaction rate goes to ~~0~~ zero. As more & more time passes, the slower the reaction rate gets. This makes sense because as t increases, the concentration of A & B decreases and A molecules & B molecules are less likely to meet.

ex 26: Suppose that a bacteria population starts w/ 500 bacteria and triples every hour.

Q: What is the population after 3 hours? After 4 hours? After t hours? What's the population doing as $t \to \infty$?

Q: Using $\frac{d}{dx}(3^x) = \ln(3)\, 3^x$, estimate the rate of increase of the population after 6 hours.

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know population triples every hour

$t=0$ Pop $= 500$

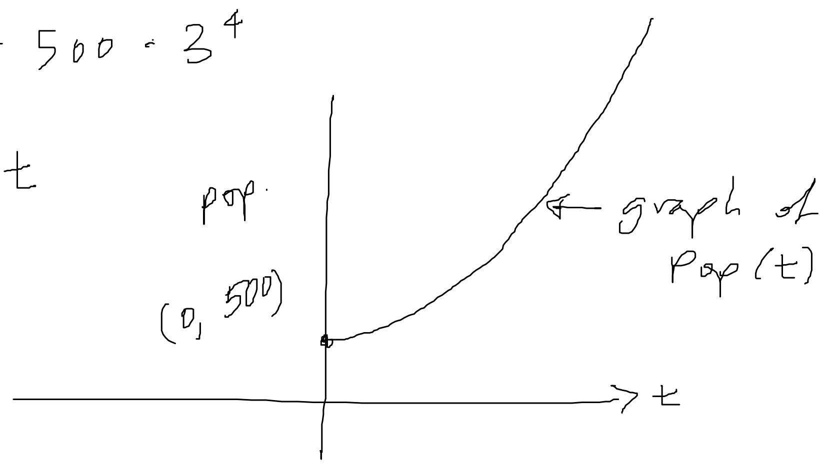
$t=1$ Pop $= 500 \cdot 3$

$t=2$ Pop $= 500 \cdot 3 \cdot 3$

$t=3$ Pop $= 500 \cdot 3 \cdot 3 \cdot 3 = 500\, 3^3$

$t=4$ pop $= 500 \cdot 3^4$

$$\text{Pop}(t) = 500 \cdot 3^t$$



as $t \to \infty$, $\text{Pop}(t) \to \infty$. Malthus would be proud.

rate of population increase?

$$\frac{d}{dt} Pop(t) = \frac{d}{dt} 500 \cdot 3^t$$

$$= 500 \frac{d}{dt} 3^t = 500 \cdot \ln(3) \cdot 3^t$$

the rate of increase is growing exponentially fast. yikes! But it makes sense since bacteria reproduce by splitting --- the more there are, the more there will be.

rate of increase after 6 hours?

$$\frac{dPop}{dt}(6) = 500 \cdot \ln(3) \cdot 3^6$$

$$\cong 400444 \text{ bacteria/hour}$$

ex: 31 If $p(x)$ is the total value of the production when there are x workers in a plant then the average productivity of the workforce at the plant is

$$A(x) = \frac{p(x)}{x}$$

Q: what's the company's goal?

Q: find $A'(x)$. What should the company do if $A'(x) > 0$? If $A'(x) < 0$?

Q: Show that $A'(x) > 0$ if $p'(x)$ is greater than the average productivity

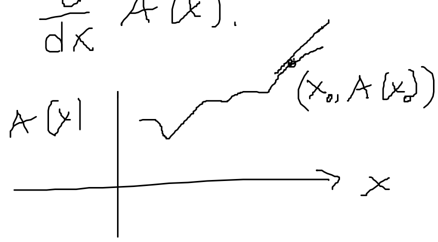
Q1: what's the company's goal?

Answer 1: maximize average productivity

Answer 2: maximize profit.

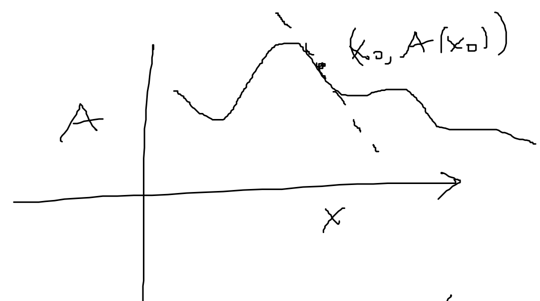
Let's look at answer 1 first. If the goal is to maximize average productivity then look at $\frac{d}{dx} A(x)$.

if $\frac{dA}{dx}(x_0) > 0$



then the company can increase the average productivity by hiring some more people.

if $\frac{dA}{dx}(x_0) < 0$



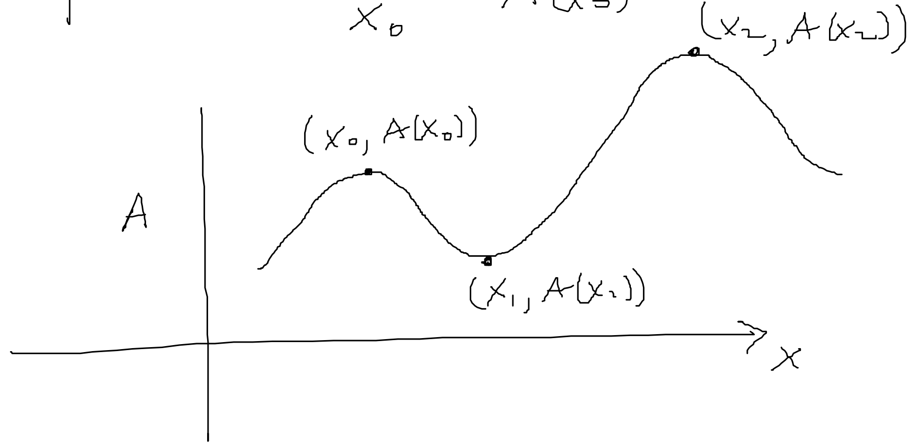
then the company can increase the average productivity by laying off some of the workers.

$$\frac{dA}{dx} = \frac{d}{dx}\frac{p(x)}{x} = \frac{x p'(x) - 1 \cdot p(x)}{x^2} = \frac{x p'(x) - p(x)}{x^2}$$

If the company seeks x_0 such that $\frac{dA}{dx}(x_0) = 0$ then it seeks x_0 such that

$$p'(x_0) = \frac{p(x_0)}{x_0} = A(x_0)$$

Note: if the graph of $A(x)$ looks like ⇒



then there would be 3 staff sizes: x_0, x_1, and x_2. But x_2 is the best staff size --- the derivative $\frac{dA}{dx}$ can't tell this though.

<u>Q2</u>: Show that $A'(x) > 0$ if $p'(x) > A(x)$.

If $p'(x) > A(x)$ then $p'(x) > \frac{p(x)}{x}$. So $x p'(x) > p(x)$ and then $x p'(x) - p(x) > 0$.

Hence $A'(x) = \frac{x p'(x) - p(x)}{x^2} > 0$. And the company should hire some more workers.

If the company's goal is to maximize profit then

$$\text{Profit} = p(x) - cx$$

where c = cost of one worker

$\Rightarrow$ cx = cost of staff

$p(x)$ = value of product.

The company wants to maximize

$$\text{Profit}(x) = p(x) - cx$$