Mat 135 Oct 18, 2004

§ 3.2 Product Rule + Quotient rule.

$$\frac{d}{dx}\left(f(x)g(x)\right) \neq \frac{df}{dx}\cdot\frac{dg}{dx}$$

If this were true then

if the wrong thing were true

$$\frac{d}{dx}\left(x^4\right) = \frac{d}{dx}\left(x^2\cdot x^2\right) \stackrel{?}{=} (2x)(2x) = 4x^2$$

$$\begin{array}{c} \| \\ 4x^3 \end{array}$$

but $4x^3 \neq 4x^2$ for all x.

(It's true for $x=1$ but not for any other values of x.)

<u>fact:</u> If f and g are both differentiable then

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}$$

for example, $f(x) = \frac{1}{x^2}$ and $g(x) = \sqrt{3-x}$

we know f is differentiable everywhere except $x=0$ at $f'(x) = \frac{-2}{x^3}$. we know $g(x)$ is differentiable on $(-\infty, 3)$ and $\frac{dg}{dx} = \frac{-1}{2\sqrt{3-x}}$ (can do this directly using definition of derivative)

And so, $\frac{\sqrt{3-x}}{x^2}$ is differentiable on $(-\infty, 0) \cup (0, 3)$

and $\frac{d}{dx}\left(\frac{\sqrt{3-x}}{x^2}\right) = \frac{d}{dx}\left(\frac{1}{x^2} \cdot \sqrt{3-x}\right)$

$$= \frac{-2}{x^3}\sqrt{3-x} + \frac{1}{x^2}\left(\frac{-1}{2\sqrt{3-x}}\right)$$

Why is the product rule true?

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} g(x+h)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x+h) - g(x)}{h}$$

$$= g(x)\frac{df}{dx} + f(x)\frac{dg}{dx}$$ as claimed.

The quotient rule:

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

for example,

$$\frac{d}{dx}\left(\frac{\sqrt{3-x}}{x^2}\right) = \frac{x^2\left(\frac{-1}{2\sqrt{3-x}}\right) - \sqrt{3-x}\,(2x)}{x^4}$$

$$= \frac{1}{x^2}\left(\frac{-1}{2\sqrt{3-x}}\right) - \frac{2}{x^3}\sqrt{3-x}$$

which is what we had before

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ex: $$\frac{d}{dt}\left(\frac{e^t}{1+t}\right) = \frac{e^t(1+t) - 1\cdot e^t}{(1+t)^2}$$

ex: $$\frac{d}{dr}\left[(r^2-2r)e^r\right] = (r^2-2r)e^r + (2r-2)e^r$$

ex: $$\frac{d}{dy}\left[(y^2+y)(\pi y^3 - y^{10})\right] = (y^2+y)(3\pi y^2 - 10y^9) + (2y+1)(\pi y^3 - y^{10})$$

ex: $\frac{d}{dx}\left(\frac{ax+b}{cx+d}\right) = \frac{a(cx+d)-c(ax+b)}{(cx+d)^2}$

ex: $f(5)=1 \quad f'(5)=6 \quad g(5)=-3 \quad g'(5)=2$

find $(fg)'(5)$.

ans: $(fg)'(5) = f(5)g'(5) + f'(5)g(5)$

$= 1\cdot 2 + 6\cdot(-3) = -16$

find $\left(\frac{f}{g}\right)'(5) = \frac{f'(5)g(5) - g'(5)f(5)}{(g(5))^2}$

$= \frac{6(-3) - (2)(1)}{(-3)^2}$

$= \frac{-18-2}{9} = \frac{-20}{9}$

find $\left(\frac{g}{f}\right)'(5) = \frac{g'(5)f(5) - f'(5)g(5)}{(f(5))^2}$

$= \frac{2(1) - 6(-3)}{1^2} = 18$

ex: if $f(x) = e^x g(x)$

where $g(0) = 2$ and $g'(0) = 5$, find $f'(0)$.

$$f'(x) = e^x g(x) + e^x g'(x)$$

$$\Rightarrow f'(0) = e^0 \cdot 2 + e^0 \cdot 5 = 7$$

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ex: if f is differentiable, find

$$\frac{d}{dx}\left(\frac{1+xf(x)}{\sqrt{x}}\right) = \frac{[xf'(x) + 1f(x)]\sqrt{x} - \frac{1}{2\sqrt{x}}(1+xf(x))}{(\sqrt{x})^2}$$

$$= \frac{x^{3/2}f'(x) + \sqrt{x}f(x) - \frac{1}{2\sqrt{x}} + \frac{1}{2}\sqrt{x}f(x)}{x}$$

$$= \frac{x^2f'(x) + xf(x) - \frac{1}{2} + \frac{1}{2}xf(x)}{x^{3/2}}$$

$$= \frac{2x^2f'(x) + 2xf(x) - 1 + xf(x)}{x^{3/2}}$$