Math 135 Oct 15, 2004

definition: A function f is differentiable at c if $f'(c)$ exists. It is differentiable on an open interval (a,b) [or $(-\infty,a)$, (a,∞), or $(-\infty,\infty)$] if it is differentiable at every point in the interval.

ex: $f(x) = x^2+1$ is defined on $(-\infty,\infty)$.
It is differentiable on $(-\infty,\infty)$

ex: $f(x) = |x+1|$ is defined on $(-\infty,\infty)$.
It is differentiable on $(-\infty,-1)$ and $(-1,\infty)$

ex: $f(x) = 2x + \frac{1}{x-4}$ is defined on $(-\infty,4)$ and $(4,\infty)$
It's differentiable on $(-\infty,4)$ and $(4,\infty)$

ex: $f(x) = \sqrt{2-x}$ is defined on $(-\infty,2]$
It's differentiable on $(-\infty,2)$.

fact: If f is differentiable on (a,b) then f is continuous on (a,b). But f continuous on (a,b) does not imply f is differentiable on (a,b). [E.g. $f(x)=|x|$ on $(-1,1)$.]

We need some rules for calculating derivatives because doing it from limits all the time is ineffiuent.

<u>rule</u>: if $f(x) = c$ (a constant function) then f is differentiable on $(-\infty, \infty)$ and $f'(x) = 0$.

<u>Why?</u> $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h} = 0.$$

<u>rule</u>: if $f(x) = x^n$ where n is a positive integer then f is differentiable on $(-\infty, \infty)$ and $f'(x) = n x^{n-1}$

<u>why?</u> First, a useful fact:

$$(x^n - a^n) = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + xa^{n-2} + a^{n-1})$$

A second useful fact:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

so $f'(a) = \lim_{x \to a} \dfrac{x^n - a^n}{x - a}$

$$= \lim_{x \to a} \frac{(x-a)(x^{n-1} + x^{n-2}a + \dots + x a^{n-2} + a^{n-1})}{(x-a)}$$

$$= \lim_{x \to a} \underbrace{[x^{n-1} + x^{n-2}a + \dots + x a^{n-2} + a^{n-1}]}_{\text{there are } n \text{ terms in this sum}}$$

$$= \underbrace{a^{n-1} + a^{n-2}a + \dots + a a^{n-2} + a^{n-1}}_{\text{there are } n \text{ terms in this sum}}$$

$$= n a^{n-1}$$

this shows $f'(a) = n a^{n-1}$ as claimed.

---

fact: if $f(x) = x^n$ and n is any real number then $f'(x) = n x^{n-1}$

---

fact: if f is differentiable and c is a constant, then cf is differentiable and $\frac{d}{dx} cf = c \frac{df}{dx}$

ex: The derivative of πx^{10} is $10\pi x^9$

Why? we know $f(x) = x^{10}$ is differentiable and by previous rule

$$\frac{d}{dx}\pi f(x) = \pi \frac{df}{dx} = \pi(10x^9) = 10\pi x^9. \checkmark$$

fact: if f and g are differentiable then $f+g$ is differentiable and

$$\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$$

ex: $\frac{d}{dx}(3x^2+9x) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(9x)$ (sum rule)

$= 3\frac{d}{dx}(x^2) + 9\frac{d}{dx}(x)$ constants rule

$= 3(2x) + 9\cdot 1 = 6x+9$

ex: $\frac{d}{dx}\left(\frac{1}{x^2} + \sqrt{3-x}\right)$?

$f(x) = \frac{1}{x^2}$ is differentiable on $(-\infty, 0)$ and $(0,\infty)$

and $\frac{df}{dx} = \frac{d}{dx}(x^{-2}) = -2x^{-3} = \frac{-2}{x^3}$

$g(x) = \sqrt{3-x}$ is differentiable on $(-\infty, 3)$

and $\dfrac{d}{dx}\sqrt{3-x} = \dfrac{d}{dx}(3-x)^{1/2}$

$$= \frac{1}{2}(3-x)^{-1/2}(-1)$$

$$= \frac{-1}{2\sqrt{3-x}}$$

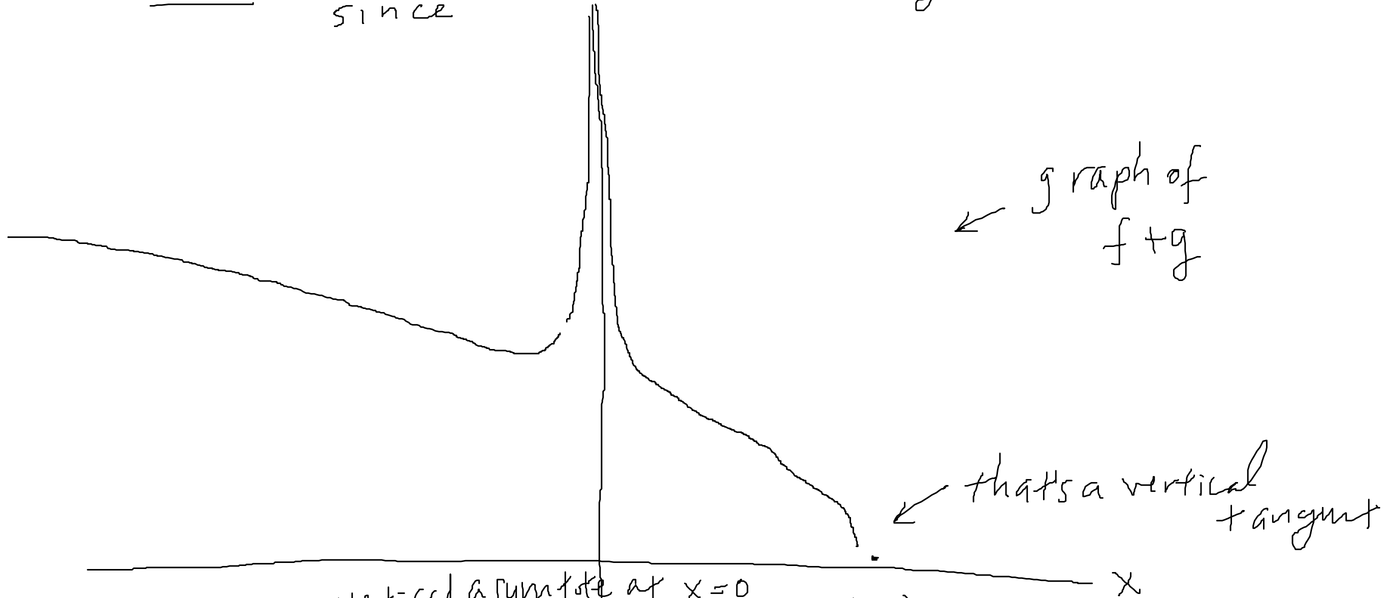
Note: I used the "chain rule" here. You'll learn this in Section 3.5

So $f+g$ is differentiable on $(-\infty, 0)$ and $(0, 3)$ and

$$\frac{d}{dx}\left(\frac{1}{x^2} + \sqrt{3-x}\right) = \frac{d}{dx}\left(\frac{1}{x^2}\right) + \frac{d}{dx}\left(\sqrt{3-x}\right)$$

$$= \frac{-2}{x^3} - \frac{1}{2\sqrt{3-x}}$$

<u>Note</u>: the derivative of $f+g$ makes sense since



Let $f(x) = b_3 x^3 + b_2 x^2 + b_1 x + b_0$

be a cubic function. What can its graph look like?

A: there are exactly 6 possibilities.

<u>Case 1</u> $b_3 > 0$

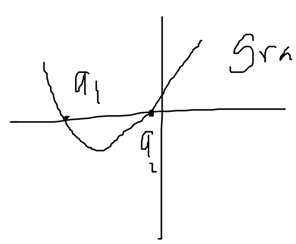
then as $x \to \infty$ $f(x) \to \infty$

and as $x \to -\infty$ $f(x) \to -\infty$

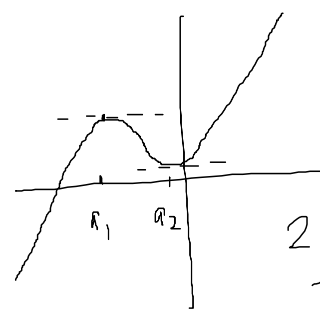
$$f'(x) = 3b_3x^2 + 2b_2x + b_1$$

since $b_3 > 0$, we see f' is an "upward" parabola

1a: $f'(x)$ has 2 zeros:

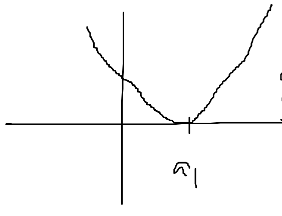


Graph of f' $\Rightarrow$

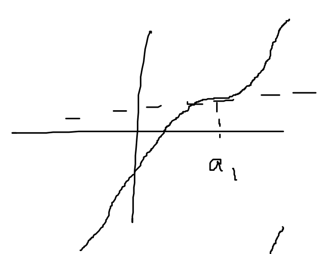


graph of f has 2 horiz. tangent

1b: $f'(x)$ has 1 zero:

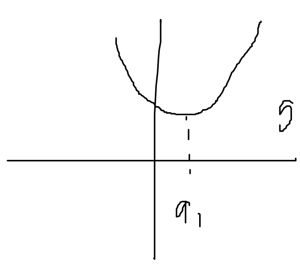


graph of f' $\Rightarrow$

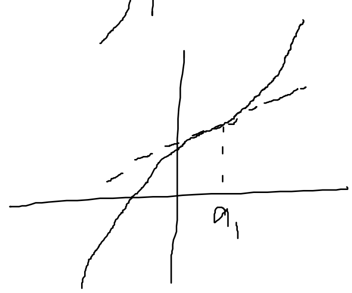


graph of f has 1 horiz. tangent

1c: $f'(x)$ has no zeros:



graph of f' $\Rightarrow$



graph of f has <u>no</u> horiz. tang.

Case 2: $b_3 < 0$

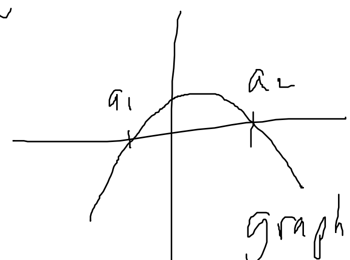
then as $x \to \infty$, $f(x) \to -\infty$

and as $x \to -\infty$ $f(x) \to +\infty$

$f'(x) = 3b_3x^2 + 2b_2x + b_1$

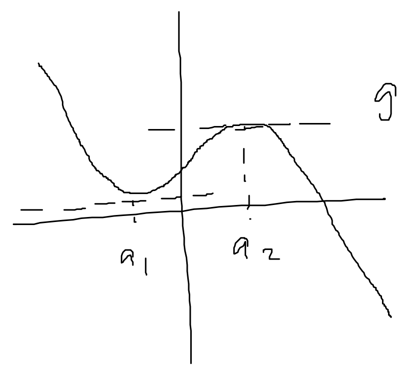
since $b_3 < 0$, f' has a "downward" parabola as its graph.

2a: f' has 2 zeros



graph of f'

$\Rightarrow$

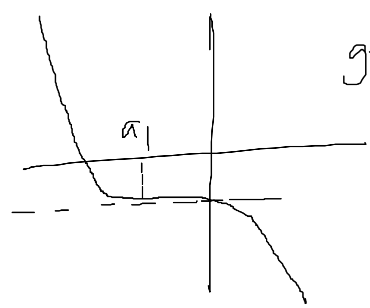


graph of f has 2 horiz. tangen

2b f' has 1 zero

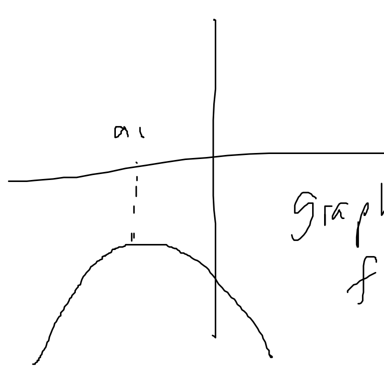
graph of f'

$\Rightarrow$



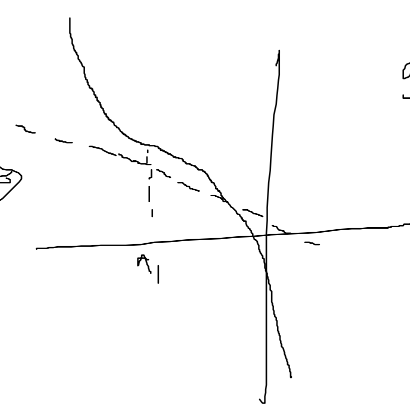
graph of f has 1 horiz. tangen

2c f' has no zeros



graph of f'

$\Rightarrow$



graph of f has no horiz. tangents

derivatives of exponential functions.

$\frac{d}{dx}(2^x) = ?$ Let $g(x) = 2^x$ then

$$g'(x) = \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$$

$$= \lim_{h \to 0} \frac{2^x 2^h - 2^x}{h} = 2^x \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$\frac{d}{dx}(3^x) = \lim_{h \to 0} \frac{3^{x+h} - 3^x}{h} = 3^x \lim_{h \to 0} \frac{3^h - 1}{h}$$

In general, $\frac{d}{dx}(a^x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$

We'd be done, if only we knew what that limit was!

$\lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69$ by calculator

$\lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.1$ by calculator

definition: e is the real number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

by calculator, $e \approx 2.72$ just as you've gotten used to π representing a particular real number, you will get used to e representing a (different) particular real number.

And so,

by how e is defined.

$$\frac{d}{dx}(e^x) = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

What about

$\frac{d}{dx}(2^x)$?

$$\frac{d}{dx}(2^x) = 2^x \lim_{h \to 0} \frac{2^h - 1}{h}$$

$$= 2^x \ln(2)$$

[We'll learn later that $\lim_{h \to 0} \frac{a^h - 1}{h} = \ln(a)$]

ex:

$$\frac{d}{dt}\left(\sqrt[3]{t^2} + 2\sqrt{t^3}\right)$$

$$= \frac{d}{dt}\left(t^{2/3} + 2t^{3/2}\right)$$

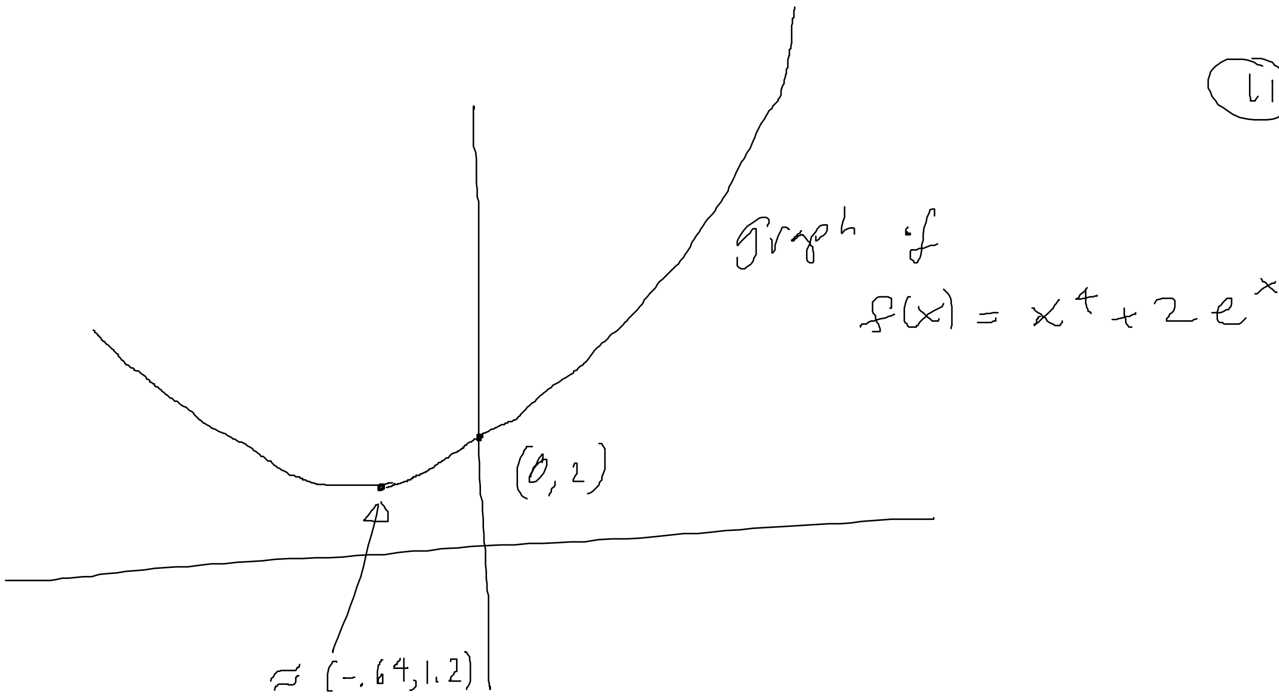
$$= \frac{d}{dt}\left(t^{2/3}\right) + \frac{d}{dt}\left(2t^{3/2}\right) \qquad \text{(sum rule)}$$

$$= \frac{d}{dt}\left(t^{2/3}\right) + 2\frac{d}{dt}\left(t^{3/2}\right) \qquad \text{(constants rule)}$$

$$= \frac{2}{3}t^{-1/3} + 2\left(\frac{3}{2}t^{1/2}\right) = \frac{2}{3}t^{-1/3} + 3t^{1/2}$$

ex: find an equation of the tangent line to the curve at the given point

$y = x^4 + 2e^x \qquad (0, 2)$



as $x \to \infty$, $f(x)$ grows like $2e^x$

as $x \to -\infty$, $f(x)$ grows like x^4

so the graph goes up to ∞ at both ends, but it goes much faster as $x \to \infty$ than as $x \to -\infty$.

tangent line at $(0,2)$

is $y = f'(0)(x-0) + 2$

$$\frac{d}{dx} f(x) = \frac{d}{dx}\left(x^4 + 2e^x\right) = 4x^3 + 2e^x$$

$$\Rightarrow f'(0) = 4(0)^3 + 2e^0 = 2$$

$\Rightarrow$ tangent line is $y = 2(x-0) + 2$

$$\boxed{y = 2x + 2}$$