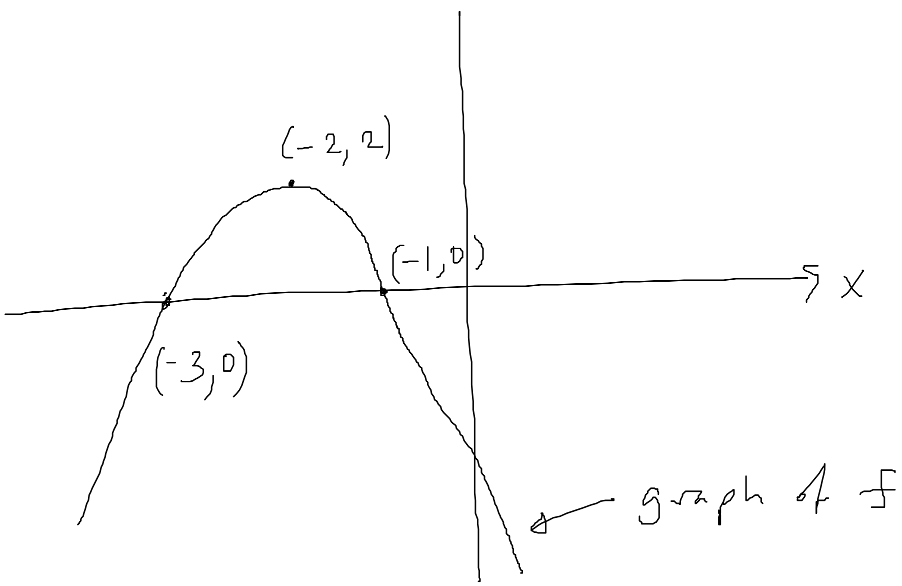
Mat 135, Oct 13 2004

Consider $f(x) = -2(x+2)^2 + 2$

$$= -2x^2 - 8x - 6 = -2(x+3)(x+1)$$



from the graph, we expect that

$f'(a) > 0$ for $a < -2$

$f'(-2) = 0$

$f'(a) < 0$ for $a > -2$

let's see if that's true!

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{[-2(a+h)^2 - 8(a+h) - 6] - [-2a^2 - 8a - 6]}{h}$$

$$= \lim_{h\to 0} \frac{[-2a^2 - 4ah - 2h^2 - 8a - 8h - 6] - [-2a^2 - 8a - 6]}{h}$$

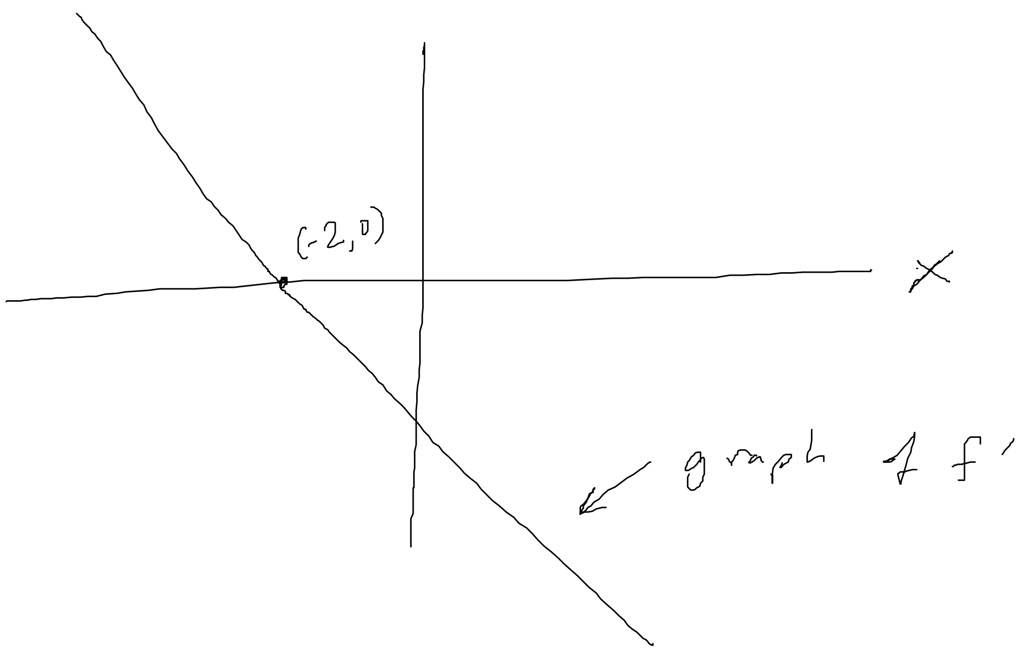
$$f'(a) = \lim_{h \to 0} \frac{-4ah - 2h^2 - 8h}{h}$$

notice that all the terms in the numerator that didn't depend on h cancelled! ☺

$$= \lim_{h \to 0} -4a - 2h - 8$$

$$= -4a - 8$$

So $f'(a) = -4a - 8$



note that all our predictions were correct!

$f'(x) > 0$ for $x < -2$ ✓

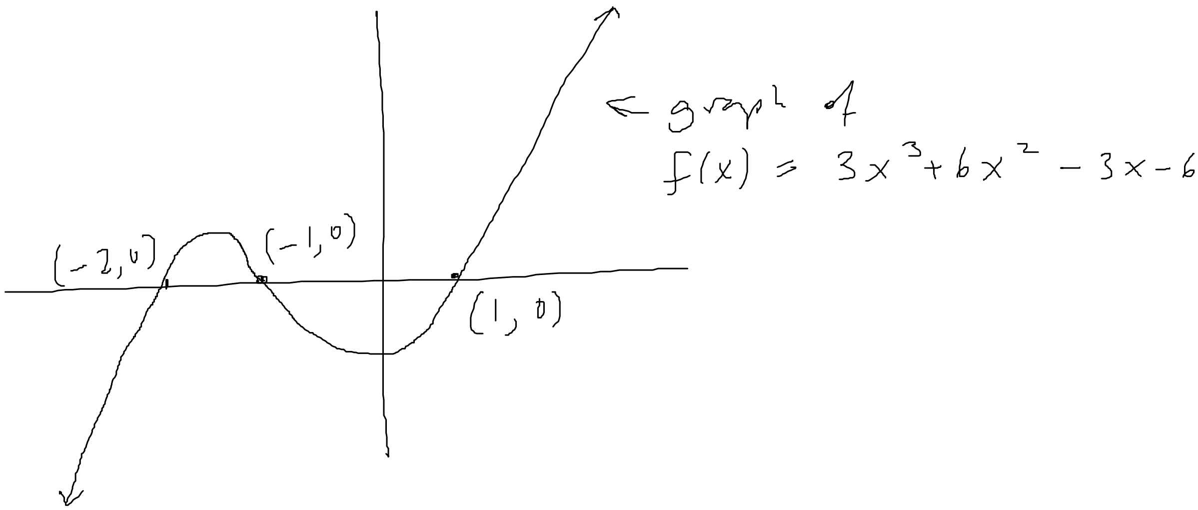
$f'(x) = 0$ at $x = -2$ ✓

$f'(x) < 0$ for $x > -2$ ✓

Consider

$$f(x) = 3(x+2)(x+1)(x-1)$$

$$= 3x^3 + 6x^2 - 3x - 6$$



we predict there are two points a_1 and a_2 where $f' = 0$. a_1 is between -2 and -1

a_2 is between -1 and 1

$f'(a) > 0$ for $a < a_1$

$f'(a_1) = 0$

$f'(a) < 0$ for $a_1 < a < a_2$

$f'(a_2) = 0$

$f'(a) > 0$ for $a > a_2$

specifically, $f'(-2) > 0$, $f'(-1) < 0$, $f'(1) > 0$

Let's check if this is true!

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{[3(a+h)^3 + 6(a+h)^2 - 3(a+h) - 6] - [3a^2 + 6a^3 - 3a - 6]}{h}$$

$$= \lim_{h \to 0} \frac{[3a^3 + 9a^2h + 9ah^2 + 3h^3 + 6a^2 + 12ah + 6h^2 - 3a - 3h - 6] - [3a^2 + 6a^2 - 3a - 6]}{h}$$

$$= \lim_{h \to 0} \frac{9a^2h + 9ah^2 + 3h^3 + 12ah + 6h^2 - 3h}{h}$$

$$= \lim_{h \to 0} 9a^2 + 9ah + 3h^2 + 12a + 6h - 3 = 9a^2 + 12a - 3$$

So $f'(a) = 9a^2 + 12a - 3$

$$= 9\left(a - \left(-\frac{2}{3} - \frac{\sqrt{7}}{3}\right)\right)\left(a - \left(-\frac{2}{3} + \frac{\sqrt{7}}{3}\right)\right)$$

$$= 9(a - a_1)(a - a_2)$$

where

$$a_1 = -\frac{2}{3} - \frac{\sqrt{7}}{3} \cong -1.54$$

$$a_2 = -\frac{2}{3} + \frac{\sqrt{7}}{3} \cong 0.22$$

So we've found $f'(a)$. It's true that:

$f'(a) > 0$ for $a < a_1$

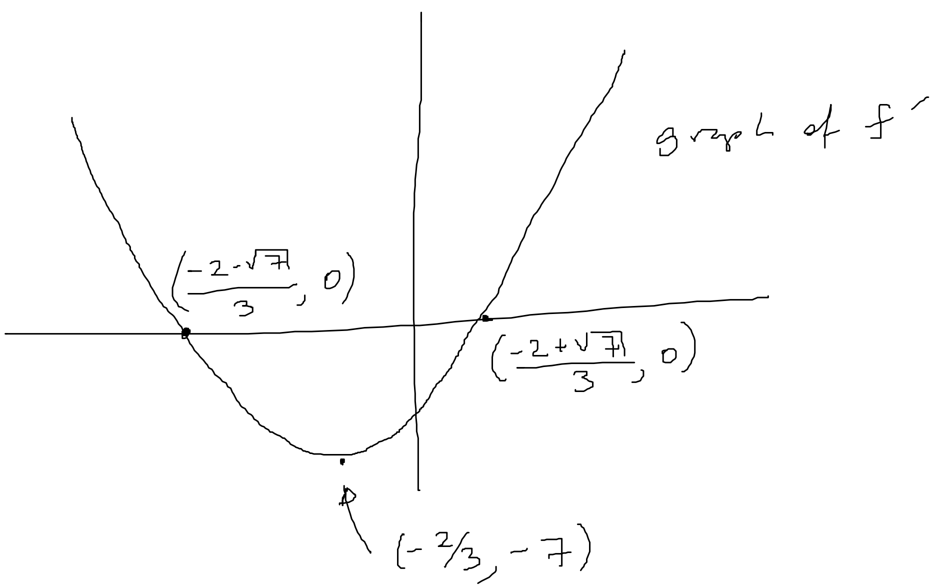
$f'(a) < 0$ for $a_1 < a < a_2$

$f'(a) > 0$ for $a > a_2$

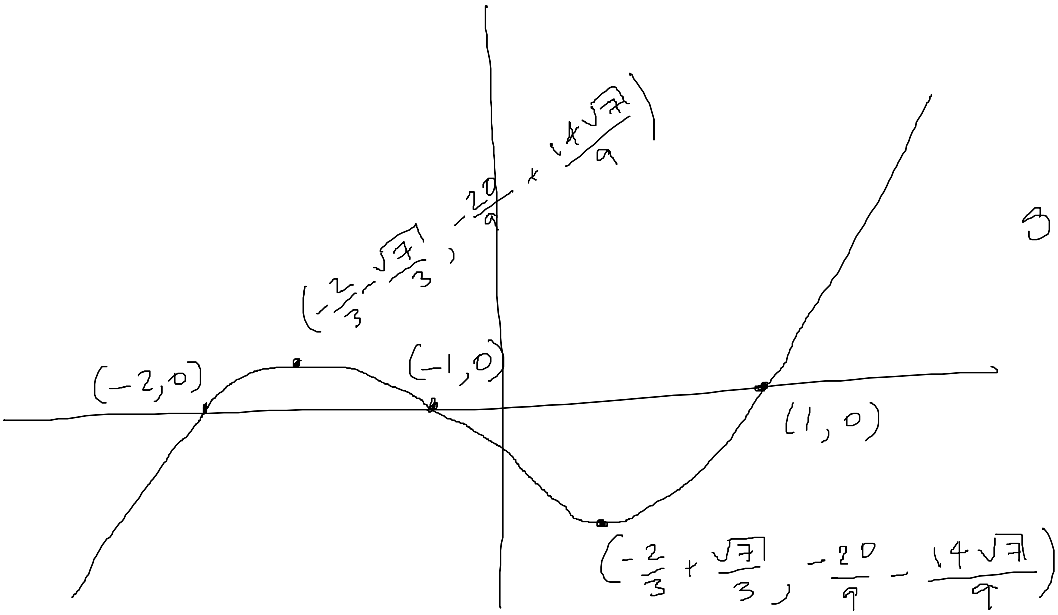
and a_1 is between -2 and -1, as expected

a_2 is between -1 and 1, as expected.

<u>Note:</u> $f'(-2) > 0$, $f'(-1) < 0$, $f'(1) > 0$, as expected.



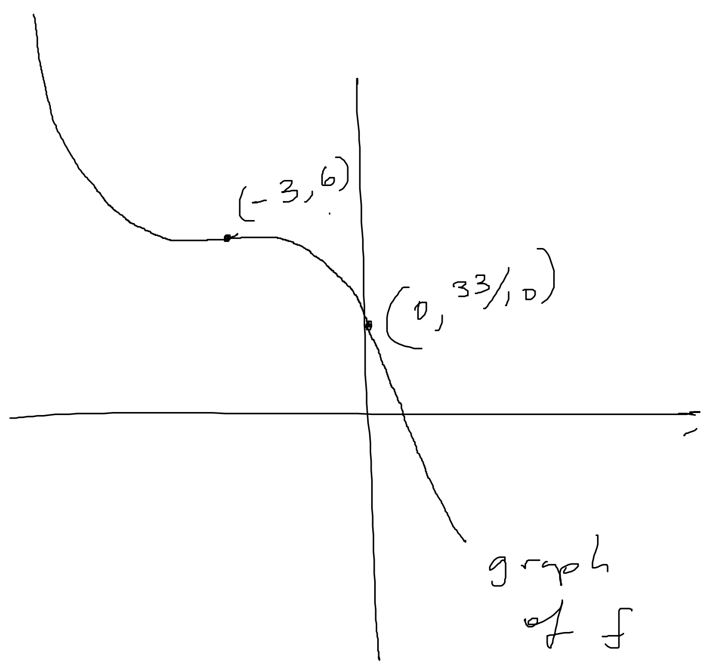
Now that we know a_1 & a_2, we can label two more points on our graph of f!



graph of f

Another example

$$f(x) = -\frac{1}{10}(x+3)^3 + 6 = -\frac{1}{10}x^3 - \frac{9}{10}x^2 - \frac{27}{10}x + \frac{33}{10}$$



graph of f

expect $f'(-3)=0$ $\quad f'(a)<0$ for $a<-3$ $\quad f'(a)<0$ for $a>-3$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \lim_{h\to 0} \frac{\left[-\frac{1}{10}(a+h)^3 - \frac{9}{10}(a+h)^2 - \frac{27}{10}(a+h) + \frac{33}{10}\right] - \left[-\frac{1}{10}a^3 - \frac{9}{10}a^2 - \frac{27}{10}a + \frac{33}{10}\right]}{h}$$

$$= \lim_{h\to 0} \frac{\left[-\frac{1}{10}a^3 - \frac{3}{10}a^2h - \frac{3}{10}ah^2 - \frac{1}{10}h^3 - \frac{9}{10}a^2 - \frac{18}{10}ah - \frac{9}{10}h^2 - \frac{27}{10}a - \frac{27}{10}h + \frac{33}{10}\right] - \left[-\frac{1}{10}a^3 - \frac{9}{10}a^2 - \frac{27}{10}a + \frac{33}{10}\right]}{h}$$

$$= \lim_{h\to 0} \frac{-\frac{3}{10}a^2h - \frac{3}{10}ah^2 - \frac{1}{10}h^3 - \frac{18}{10}ah - \frac{9}{10}h^2 - \frac{27}{10}h}{h}$$

$$= \lim_{h\to 0} -\frac{3}{10}a^2 - \frac{3}{10}ah - \frac{1}{10}h^2 - \frac{18}{10}a - \frac{9}{10}h - \frac{27}{10}$$

$$= -\frac{3}{10}a^2 - \frac{18}{10}a - \frac{27}{10} = -\frac{3}{10}\left[a^2 + 6a + 9\right] = -\frac{3}{10}(a+3)^2$$

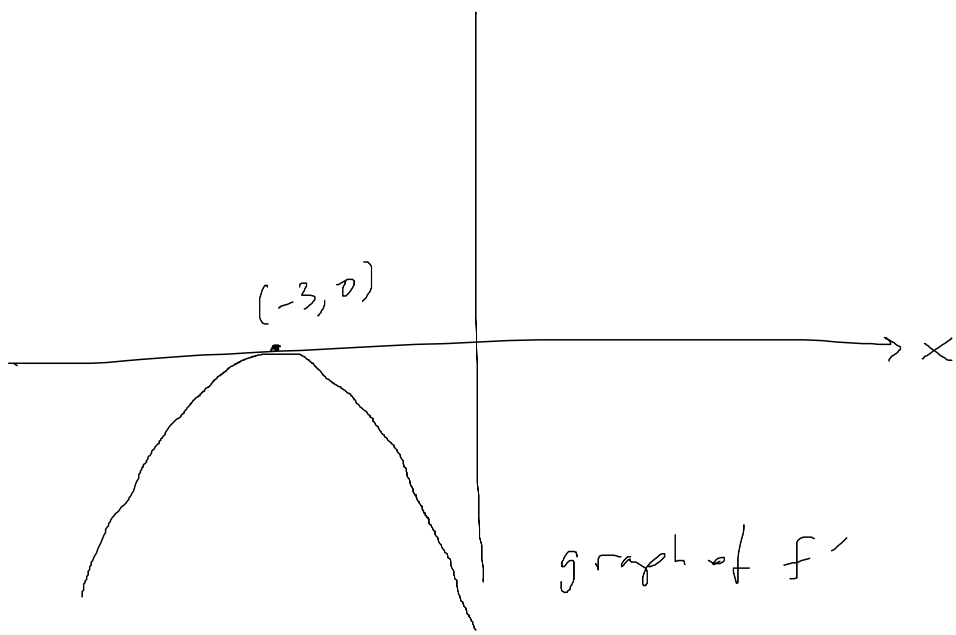
This shows that

$$f'(a) = -\frac{3}{10}(a+3)^2$$

So $f'(a) < 0$ for $a < -3$ as expected.

$f'(a) = 0$ for $a = -3$ as expected.

$f'(a) < 0$ for $a > -3$ as expected.



graph of f'

Note: In all three examples, I gave f initially in a "nice" form so that we could easily graph f. In general, you'll be given f as an expanded polynomial, and you'll use calculus to determine the graph of f. Specifically, you'd be given

$f(x) = -\frac{1}{10}x^3 - \frac{9}{10}x^2 - \frac{27}{10}x + \frac{33}{10}$ and would have to deduce its graph.