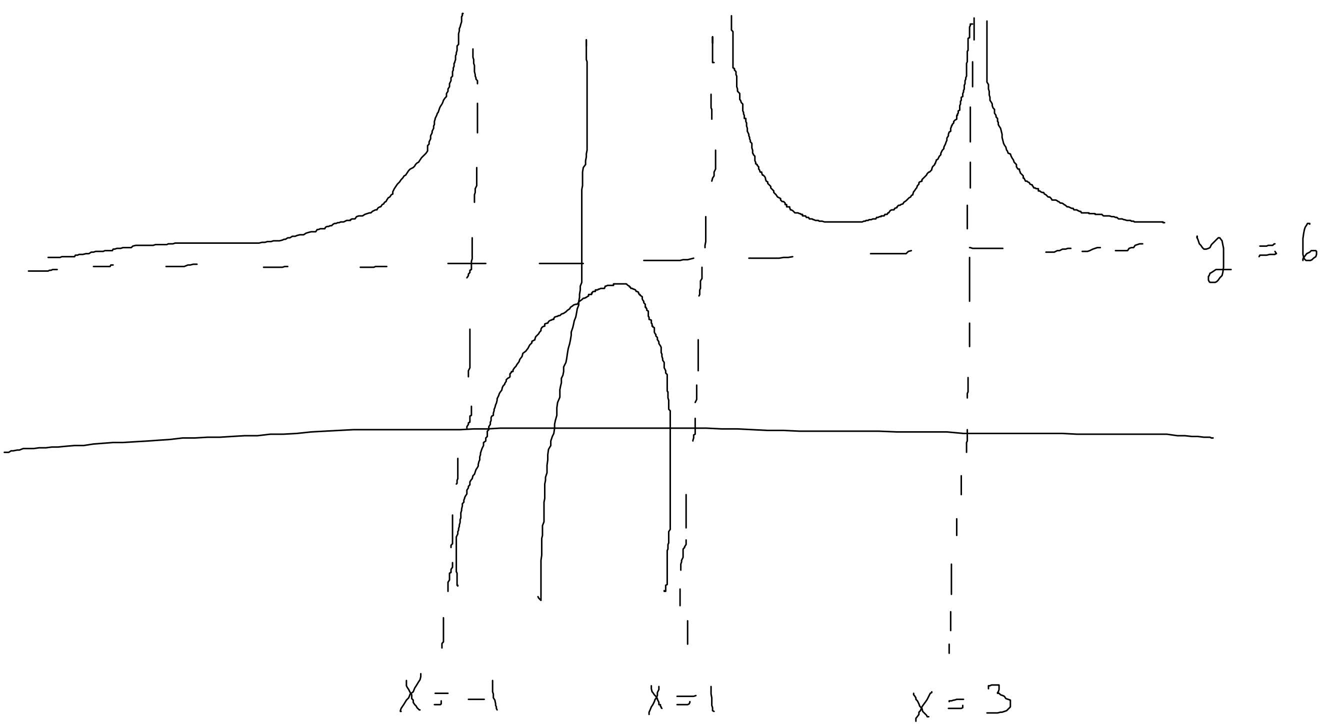
Mat 135, Oct 1, 2004

§2.6 Limits at Infinity; Horizontal Asymptotes.

consider

$$f(x) = 6 + \frac{2}{(x-1)(x+1)(x-3)^2}$$

You're fine with the graph of f being



and $\lim_{x\to\infty} f(x) = 6$ $\lim_{x\to-\infty} f(x) = 6$ seems reasonable.

x	$f(x)$	x	$f(x)$
10	~ 6.000412286	-10	~ 6.000119539
100	~ 6.000000021	-100	~ 6.000000019
1000	~ 6.000000010	-1000	~ 6.000000000

that is, by the time $x \geq 1{,}000$ $f(x)$ agrees with 6 up to 9 decimal places.

Similarly, by the time $x \leq -1{,}000$ $f(x)$ agrees with 6 up to 9 decimal places.

What about trickier examples?

$$f(x) = \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$$ for example.

option 1: $$f(x) = \frac{x^2(3 - 1/x + 4/x^2)}{x^2(2 + 5/x - 8/x^2)}$$

so $$f(x) = \begin{cases} \dfrac{3 - 1/x + 4/x^2}{2 + 5/x - 8/x^2} & \text{if } x \neq 0 \\ -2 & \text{if } x = 0 \end{cases}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3 - 1/x + 4/x^2}{2 + 5/x - 8/x^2}$$

since the fact that $f(x)$ and $g(x) = \dfrac{3 - 1/x + 4/x^2}{2 + 5/x - 8/x^2}$ disagree at $x = 0$ doesn't affect the limit as $x \to \infty$

$$= \frac{3}{2}$$

similarly,

$$\lim_{x\to-\infty} f(x) = \frac{3}{2}$$

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option 2: use long division

$$\begin{array}{r} \frac{3}{2} \\ 2x^2+5x-8 \overline{\smash{)}\, 3x^2 - x + 4} \\ 3x^2 + \frac{15}{2}x - 12 \\ \hline -\frac{17}{2}x + 16 \end{array}$$

So $f(x) = \dfrac{3}{2} + \dfrac{-\frac{17}{2}x + 16}{2x^2 + 5x - 8}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3}{2} + \frac{-\frac{17}{2}x + 16}{2x^2+5x-8} = \frac{3}{2} + \lim_{x\to\infty} \frac{-17x + 16}{2x^2+5x-8} = \frac{3}{2}$$

Admittedly, the long division looks like too much work at the moment, but you'll find it's useful in the future

ex:

$$\lim_{t\to\infty} \frac{t^2+2}{t^3+t-1} = \lim_{t\to\infty} \frac{t^3(1/t + 2/t^2)}{t^3(1+1/t^2 - 1/t^3)}$$

$$= \lim_{t\to\infty} \frac{1/t + 2/t^2}{1 + 1/t^2 - 1/t^3} = 0$$

ex:

$$\lim_{x\to\infty} \frac{\sqrt{9x^6 - x}}{x^3+1} = \lim_{x\to\infty} \frac{\sqrt{x^6(9 - 1/x^5)}}{x^3(1+1/x^3)}$$

$$= \lim_{x\to\infty} \frac{x^3\sqrt{9 - 1/x^5}}{x^3(1+1/x^3)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{9 - 1/x^5}}{1 + 1/x^3} = \sqrt{9} = 3$$

ex: $\lim_{x \to -\infty} \sqrt{9x^2 + x} - 3x$?

$\lim_{x \to \infty} \sqrt{9x^2 + x} - 3x$?

Wrong answer!

$$\lim_{x \to \infty} \sqrt{9x^2 + x} = \infty$$

$$\lim_{x \to \infty} -3x = -\infty$$

so ... $\lim_{x \to \infty} \sqrt{9x^2 + x} - 3x = \infty + (-\infty)$
$= 0.$

this is wrong wrong wrong!

$$\lim_{x \to \infty} \sqrt{9x^2 + x} - 3x$$

$$= \lim_{x \to \infty} \sqrt{9x^2(1 + 1/9x)} - 3x$$

$$= \lim_{x \to \infty} 3x\sqrt{1 + 1/9x} - 3x$$

$$= \lim_{x \to \infty} 3x\left[\sqrt{1 + 1/9x} - 1\right]$$

Note that the

$3x$ term goes to ∞ and

the $\left[\sqrt{1+\frac{1}{9x}} - 1\right]$ term goes to 0 as $x \to \infty$

so something interesting could be happening.

$$\lim_{x\to\infty} \left\{\sqrt{9x^2+x} - 3x\right\} \frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{(9x^2+x)-9x^2}{\sqrt{9x^2+x}+3x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x}{x\left[\sqrt{9+1/x}+3\right]}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{9+1/x}+3} = \frac{1}{6}$$

$$\lim_{x\to-\infty} \sqrt{9x^2+x} - 3x = \lim_{x\to-\infty} \sqrt{9x^2}\sqrt{1+\frac{1}{9x}} - 3x$$

$$= \lim_{x\to-\infty} |3x|\sqrt{1+1/9x} - 3x$$

$$= \lim_{x\to-\infty} -3x\sqrt{1+1/9x} - 3x = \lim_{x\to-\infty} -3x\left\{\left[\sqrt{1+\frac{1}{9x}}\right]+1\right\}$$

so $\lim_{x \to -\infty} \sqrt{9x^2+x} - 3x =$

$$\lim_{x \to -\infty} -3x \left[\sqrt{1+ 1/9x} + 1 \right]$$

$-3x \to +\infty$ as $x \to -\infty$

$\sqrt{1+\frac{1}{9x}} + 1 \to 2$ as $x \to -\infty$

so $\lim_{x \to -\infty} \sqrt{9x^2+x} - 3x = \infty.$

<u>ex:</u>

$$\lim_{x \to \infty} \frac{x^3-2x+3}{5-2x^2} = \lim_{x \to \infty} \frac{x^2 \left(x - 2/x + 3/x^2 \right)}{x^2 \left(-2 + 5/x^2 \right)}$$

$$= \lim_{x \to \infty} \frac{x - 2/x + 3/x^2}{-2 + 5/x^2}$$

numer $\to \infty$
denom $\to -2$ $\Rightarrow \lim_{x \to \infty} \frac{x^3-2x+3}{5-2x^2} = -\infty$

similarly, $\lim_{x \to -\infty} \frac{x^3-2x+3}{5-2x^2} = \infty.$