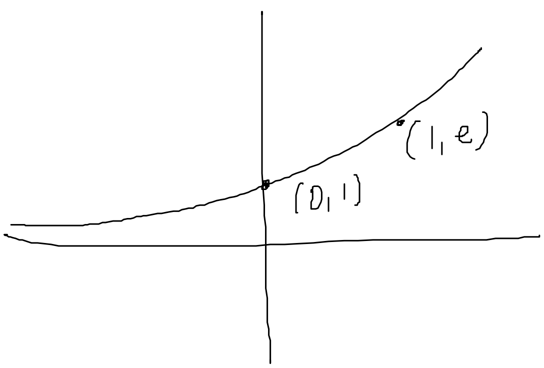
Mat 135, Nov 3 2004

Recall that ln is the inverse function of exp

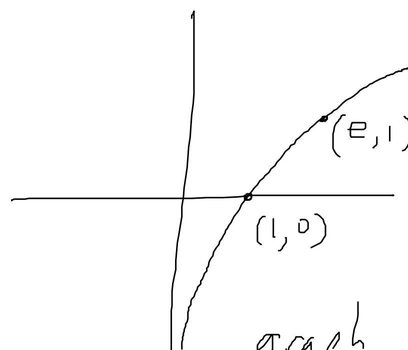
$\exp(x) = e^x$

$\exp : (-\infty, \infty) \to (0, \infty)$

$\Rightarrow \ln : (0, \infty) \to (-\infty, \infty)$



graph of exp $\Rightarrow$



graph of ln.

We find the derivative of ln by implicit differentiation

We know $\exp(\ln(x)) = x$ for all x in $(0, \infty)$

$\Rightarrow \frac{d}{dx} \exp(\ln(x)) = \frac{d}{dx}(x) = 1$

$\Rightarrow \frac{d}{dx} e^{\ln(x)} = 1 \Rightarrow e^{\ln(x)} \cdot \frac{d}{dx} \ln(x) = 1$

$\Rightarrow \frac{d}{dx} \ln(x) = \frac{1}{e^{\ln(x)}} = \frac{1}{x}$

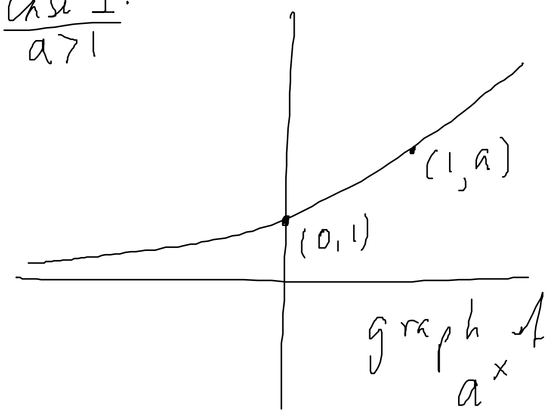
Similarly, if $a > 0$ and $a \neq 1$

then $\log_a(x)$ is the inverse of a^x

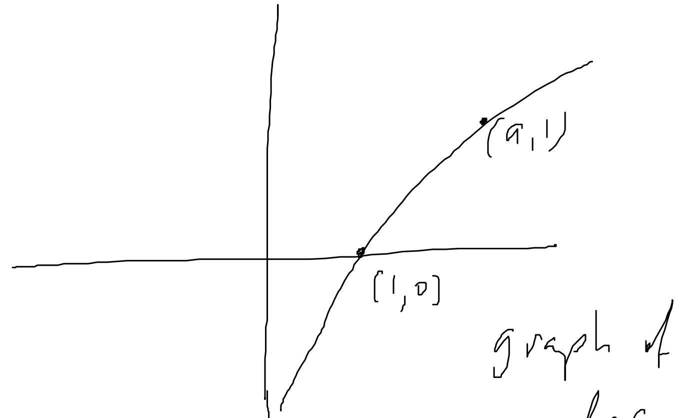
$a^x : (-\infty, \infty) \to (0, \infty)$

so $\log_a : (0, \infty) \to (-\infty, \infty)$

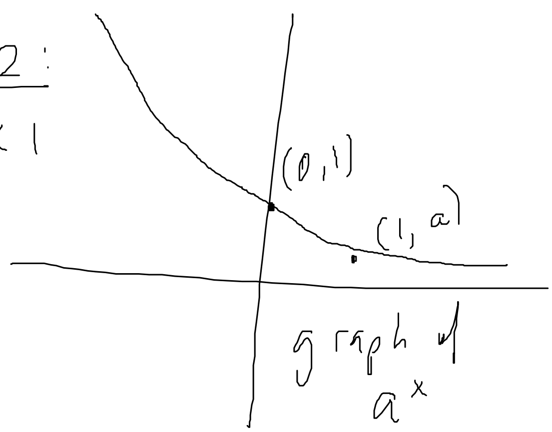
case 1:
$a > 1$



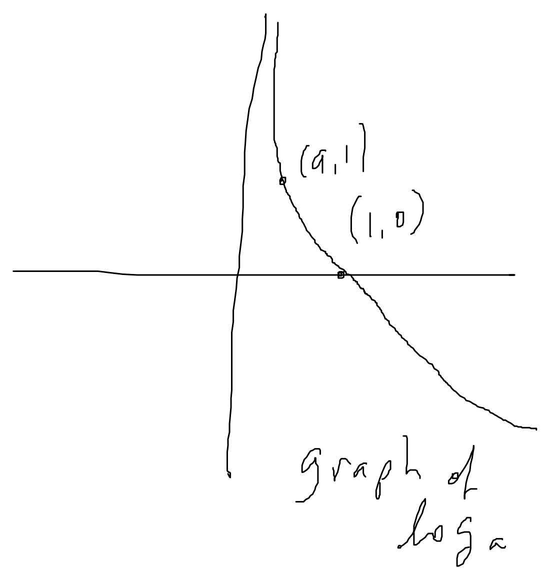
$\Rightarrow$



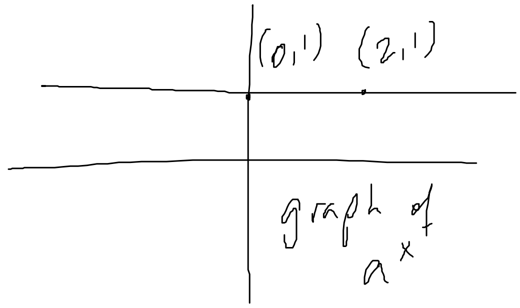
case 2:
$0 < a < 1$



$\Rightarrow$



case 3:
$a = 1$



$\Rightarrow$ no inverse! not one to one.

We find the derivative of $\log_a$ by implicit differentiation:

$$\frac{d}{dx} a^{\log_a(x)} = \frac{d}{dx} x$$

recall $a^y = \left(e^{\ln a}\right)^y = e^{(\ln a)\cdot y}$

$$\Rightarrow \frac{d}{dx} e^{(\ln a)\log_a(x)} = 1$$

$$\Rightarrow e^{(\ln a)\log_a(x)} \cdot \ln(a) \frac{d}{dx} \log_a(x) = 1$$

$$\Rightarrow a^{\log_a(x)} \ln(a) \frac{d}{dx} \log_a(x) = 1$$

$$\Rightarrow x \ln(a) \frac{d}{dx} \log_a(x) = 1$$

since $a^{\log_a(x)} = x$

$$\Rightarrow \frac{d}{dx} \log_a(x) = \frac{1}{\ln(a)\, x}$$

Summing up:

$$\boxed{\begin{aligned} \frac{d}{dx} \ln(x) &= \frac{1}{x} \\ \frac{d}{dx} \log_a(x) &= \frac{1}{\ln(a)\, x} \end{aligned}}$$

ex # 10 differentiate the function

$$f(t) = \frac{1+\ln(t)}{1-\ln(t)}$$

$$\frac{d}{dt}\frac{1+\ln(t)}{1-\ln(t)} = \frac{\left[\frac{d}{dt}(1+\ln(t))\right](1-\ln(t)) - (1+\ln(t))\frac{d}{dt}(1-\ln(t))}{(1-\ln(t))^2}$$

$$= \frac{\frac{1}{t}(1-\ln(t)) - (1+\ln(t))(-\frac{1}{t})}{(1-\ln(t))^2}$$

$$= \frac{\frac{1}{t} - \frac{\ln(t)}{t} + \frac{1}{t} + \frac{\ln(t)}{t}}{(1-\ln(t))^2}$$

$$= \boxed{\frac{2}{t(1-\ln(t))^2}}$$

14 $\frac{d}{dy}\ln(x^4\sin^2 x) = \frac{1}{x^4\sin^2 x}\frac{d}{dx}(x^4\sin^2 x)$

$$= \frac{1}{x^4\sin^2(x)}\left[4x^3\sin^2(x) + 2x^4\sin(x)\cos x\right]$$

$$= \frac{4}{x} + \frac{2\cos(x)}{\sin(x)} = \boxed{\frac{4}{x} + 2\cot(x)}$$

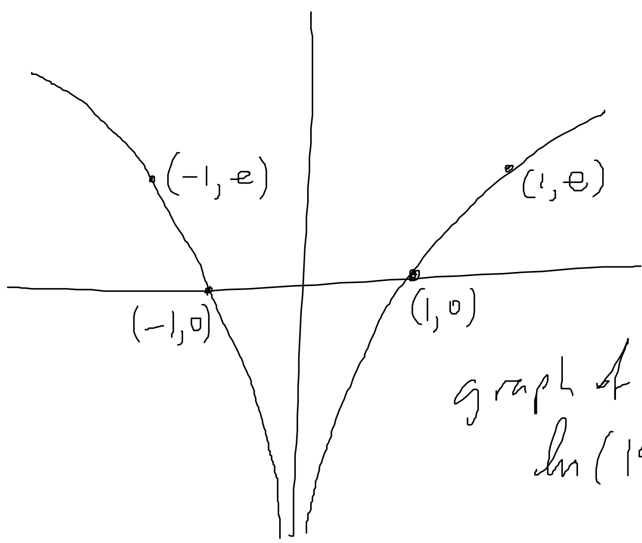
\# 17 $\frac{d}{dx} \ln|2-x-5x^2|$

Whoah! we know $\frac{d}{d\theta} \ln\theta = \frac{1}{\theta}$.

but what about

$\frac{d}{d\theta} \ln(|\theta|)$??

$\ln(|\theta|) : (-\infty,0) \cup (0,\infty) \to (-\infty,\infty)$



graph of $\ln(|\theta|)$

from the graph, we see that the derivative is < 0 if $\theta < 0$ and is > 0 if $\theta > 0$.

$$\ln(|\theta|) = \begin{cases} \ln(\theta) & \text{if } \theta > 0 \\ \ln(-\theta) & \text{if } \theta < 0 \end{cases} \Rightarrow \frac{d}{d\theta}\ln(|\theta|) = \begin{cases} \frac{1}{\theta} & \text{if } \theta > 0 \\ \frac{-1}{-\theta} & \text{if } \theta < 0 \end{cases}$$

$$\Rightarrow \frac{d}{d\theta}\ln(|\theta|) = \frac{1}{\theta}$$

And so,

$$\frac{d}{dx} \ln|2-x-5x^2| = \frac{1}{2-x-5x^2}(-1-10x)$$

$$= \boxed{-\frac{1+10x}{2-x-5x^2}}$$

\# 35 $\frac{d}{dx}(2x+1)^5(x^4-3)^6$

way 1: product rule.

$$\text{derivative} = 5(2x+1)^4(2)(x^4-3)^6 + (2x+1)^5\,6(x^4-3)^5(4x^3)$$

$$= \boxed{10(2x+1)^4(x^4-3)^6 + 24x^3(2x+1)^5(x^4-3)^5}$$

way 2: logarithmic differentiation

let $y(x) = (2x+1)^5(x^4-3)^6$

then $\ln(y(x)) = 5\ln(2x+1) + 6\ln(x^4-3)$

$\Rightarrow \frac{d}{dx}\ln(y(x)) = \frac{d}{dx}($ " " " $)$

So $$\frac{1}{y(x)}\frac{dy}{dx} = \frac{5\cdot 2}{2x+1} + \frac{6}{x^4-3}(4x^3)$$

$$\Rightarrow \frac{dy}{dx} = y(x)\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$= (2x+1)^5(x^4-3)^6\left[\frac{10}{2x+1} + \frac{24x^3}{x^4-3}\right]$$

$$= \boxed{10(2x+1)^4(x^4-3)^6 + 24x^3(2x+1)^5(x^4-3)^5}$$

sometimes logarithmic differentiation is easier than doing it directly. See book.

#37 $$y(x) = \frac{\sin^2 x \tan^4 x}{(x^2+1)^2}$$

$$\Rightarrow \ln|y(x)| = 2\ln|\sin x| + 4\ln(\tan x) - 2\ln(x^2+1)$$

$$\Rightarrow \frac{1}{y(x)}\frac{dy}{dx} = \frac{2}{\sin x}\cdot\cos x + \frac{4}{\tan x}\cdot\sec^2 x - \frac{2}{x^2+1}\cdot 2x$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{\sin^2 x\tan^4 x}{(x^2+1)}\left[2\frac{\cos x}{\sin x} - 4\frac{\sec^2 x}{\tan x} - 4\frac{x}{x^2+1}\right]}$$

§ 3.9 Hyperbolic functions

$\sinh(x) := \dfrac{e^x - e^{-x}}{2}$ $\qquad$ $\cosh(x) = \dfrac{e^x + e^{-x}}{2}$

$\tanh(x) := \dfrac{\sinh(x)}{\cosh(x)}$ $\qquad$ $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$

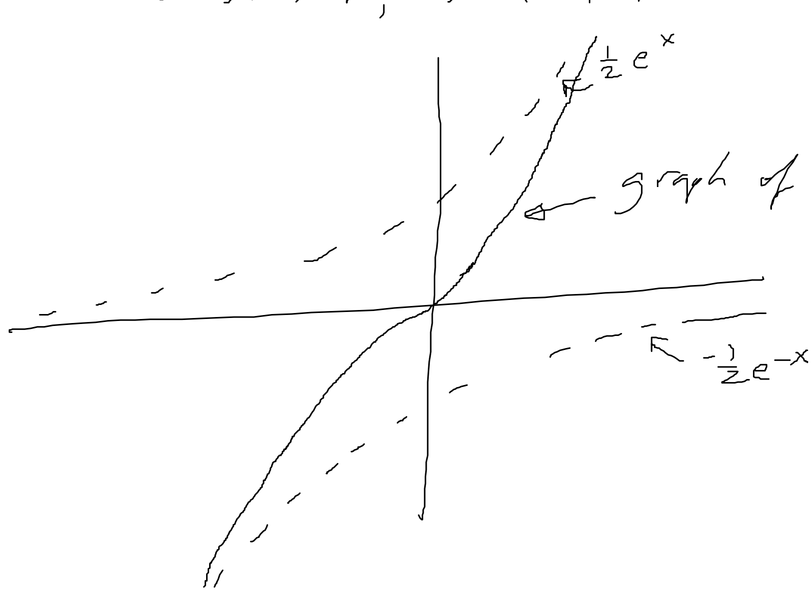
$\operatorname{sech}(x) = \dfrac{1}{\cosh(x)}$ $\qquad$ $\operatorname{csch}(x) = \dfrac{1}{\sinh(x)}$

graph of $\sinh(x)$?

$$\sinh(x) = \frac{1}{2}e^x - \frac{1}{2}e^{-x}$$

as $x \to \infty$ $\sinh(x) \to \infty$ like $\frac{1}{2}e^x$

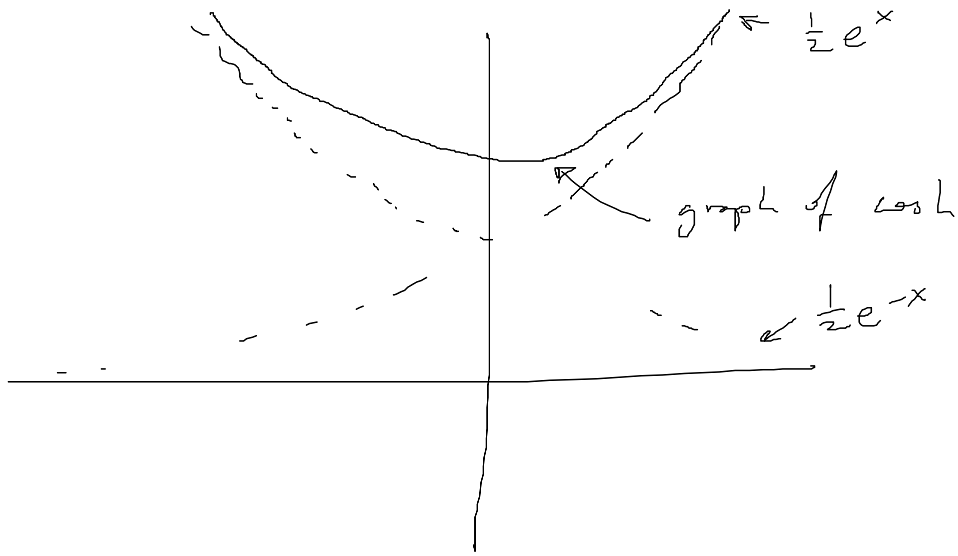
as $x \to -\infty$, $\sinh(x) \to -\infty$ like $-\frac{1}{2}e^{-x}$



$\cosh(x) = \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}$

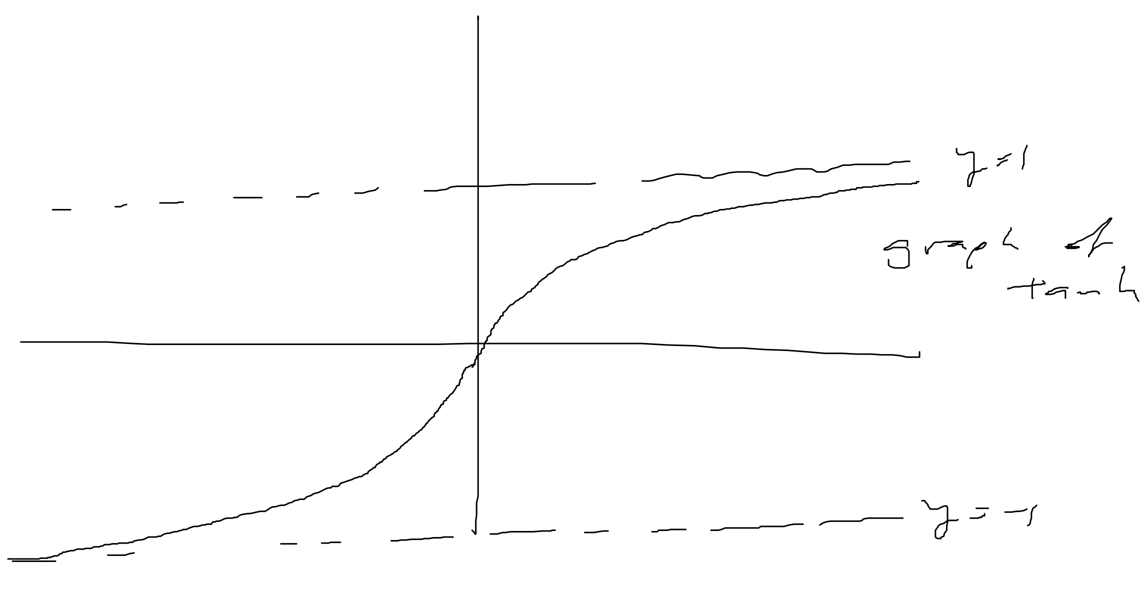
as $x \to \infty$ $\cosh(x) \to \infty$ like $\frac{1}{2}e^{x}$

as $x \to -\infty$ $\cosh(x) \to \infty$ like $\frac{1}{2}e^{-x}$



$\tanh(x)$ as $x \to \infty$ $\tanh(x) \to 1$

as $x \to -\infty$ $\tanh(x) \to -1$



$$\frac{d}{dx}\cosh(x) = \frac{d}{dx}\left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right)$$

$$= \frac{1}{2}e^{x} - \frac{1}{2}e^{-x} = \sinh(x)$$

$$\frac{d}{dx}\sinh(x) = \frac{d}{dx}\left(\frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\right)$$

$$= \frac{1}{2}e^{x} + \frac{1}{2}e^{-x} = \cosh(x)$$

$$\frac{d}{dx}\tanh(x) = \frac{d}{dx}\frac{\sinh(x)}{\cosh(x)}$$

$$= \frac{\cosh(x)\cosh(x) - \sinh(x)\sinh(x)}{\cosh^2(x)}$$

$$= \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}.$$

but $\cosh^2 - \sinh^2 = 1$ so $\frac{d}{dx}\tanh(x) = \frac{1}{\cosh^2(x)} = \operatorname{sech}^2(x).$

$$\cosh^2(x) - \sinh^2(x) = \left(\frac{1}{2}e^{x} + \frac{1}{2}e^{-x}\right)^2 - \left(\frac{1}{2}e^{x} - \frac{1}{2}e^{-x}\right)^2$$

$$= \frac{1}{4}\left(e^{2x} + 2 + e^{-2x}\right) - \frac{1}{4}\left(e^{2x} - 2 + e^{-2x}\right) = 1$$