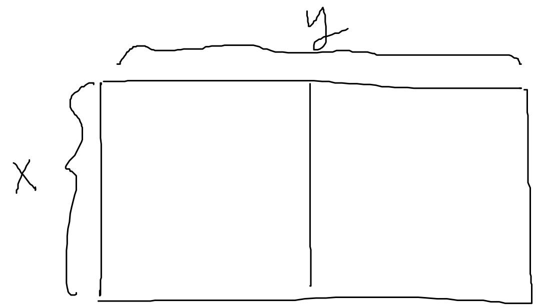
Mat 135, Nov 26 2004

§4.7 Optimization Problems!

#9 A farmer wants to fence in 1,500,000 ft^2 in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?



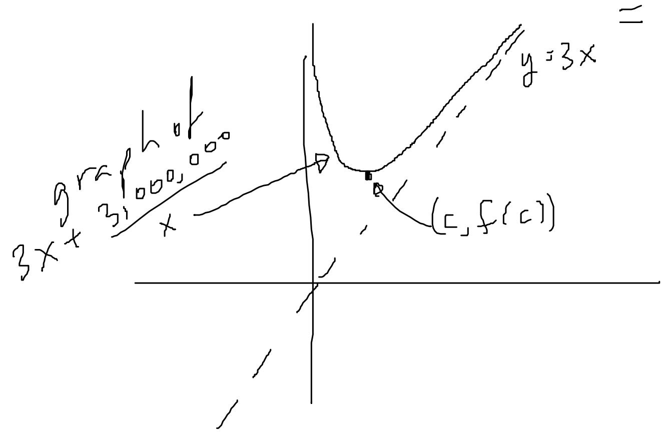
total amount of fence needed = $3x + 2y$

area enclosed = $xy = 1,500,000$

$$\Rightarrow \quad y = \frac{1,500,000}{x}$$

total amount of fence needed

$$= 3x + \frac{3,000,000}{x}$$



find the critical point of

$$f(x) = 3x + \frac{3{,}000{,}000}{x}$$

$$f'(x) = 3 - \frac{3{,}000{,}000}{x^2}$$

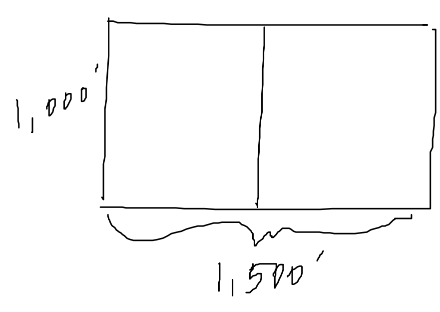
$$f'(c) = 0 \Rightarrow 3c^2 = 3{,}000{,}000$$

$$\Rightarrow c^2 = 1{,}000{,}000$$

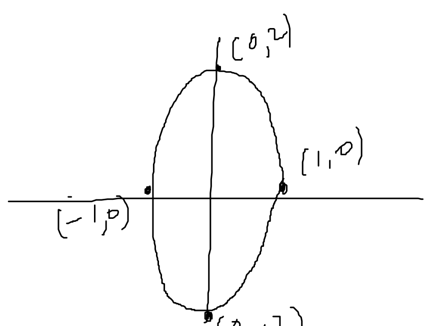
$$\Rightarrow c = 1{,}000$$

farmer needs $3(1{,}000) \times \frac{3{,}000{,}000}{1{,}000}$ feet of fence

$= 6{,}000$ feet of fence.



17 find the points on the ellipse $4x^2 + y^2 = 4$ that are furthest from the point $(1, 0)$



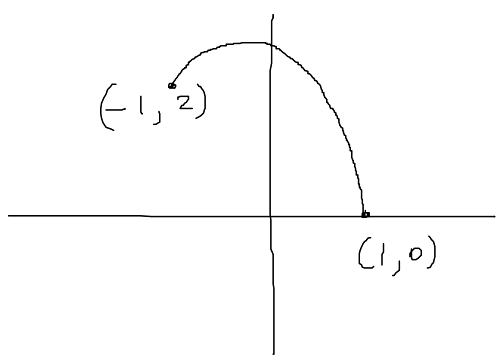
distance to $(1,0)$

$$= \sqrt{(x-1)^2 + y^2}$$

if (x,y) is on the ellipse then $y^2 = 4 - 4x^2$

$\Rightarrow$ distance to $(1,0) = \sqrt{(x-1)^2 + 4 - 4x^2}$

maximize $d(x) = \sqrt{(x-1)^2 + 4 - 4x^2} = \sqrt{-(3x+5)(x-1)}$



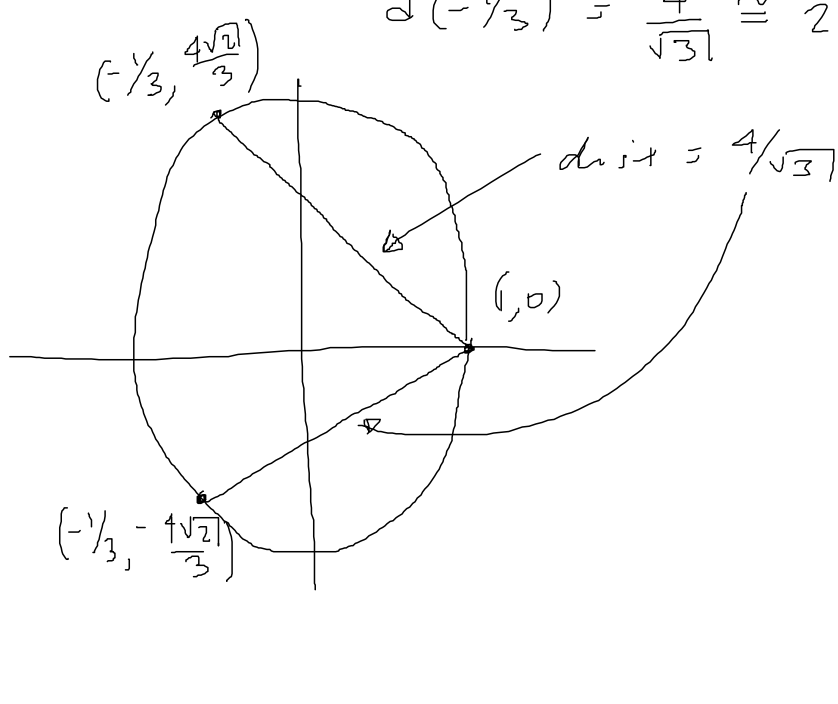
$-1 \leq x \leq 1$

since (x,y) is on the ellipse

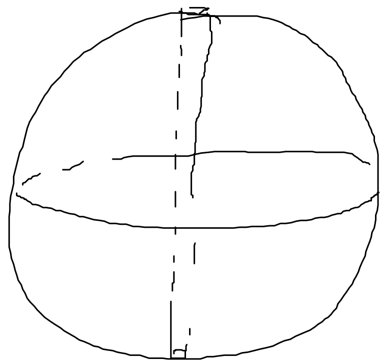
$$d'(x) = \frac{-\,3x+1}{\sqrt{-(3x+5)(x-1)}}$$

$d'(c) = 0$ at $c = -1/3$

$d(-1/3) = \frac{4}{\sqrt{3}} \cong 2.31$

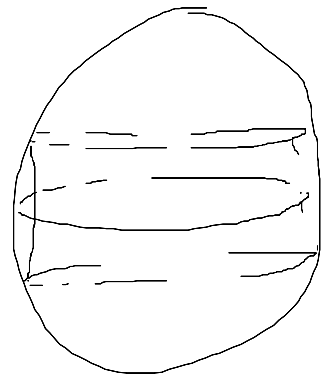


#25 A right circular cylinder is inscribed in a sphere of radius r. Find the largest possible volume of such a cylinder

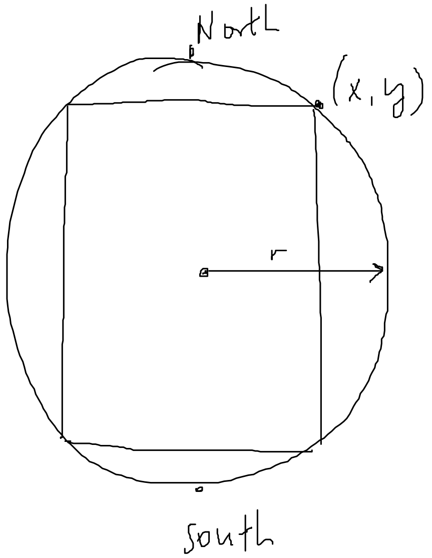


case 1 cylinder radius really small. (spaghetti)

volume of cylinder really small



case 2 cylinder really short & wide
Then volume will also be small



cut through the North pole to south pole with a plane

spin this figure to get the 3D object

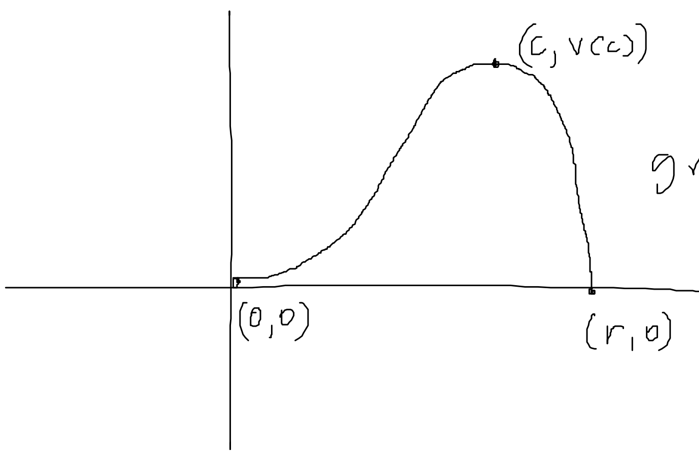
(x, y) on cylinder $\Rightarrow$ $x^2 + y^2 = r^2$ $=$ $y = \sqrt{r^2 - x^2}$

height of cylinder $= 2y = 2\sqrt{r^2 - x^2}$

radius of cylinder $= x$

And so, volume of cylinder $= \pi x^2 2y$

$$V(x) = \pi x^2 \, 2\sqrt{r^2 - x^2}$$



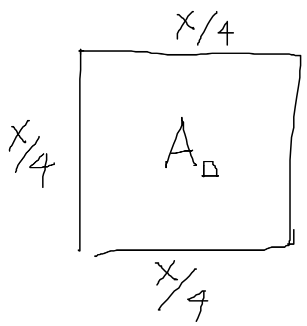
$$V'(x) = \frac{2\pi x\,(2r^2 - 3x^2)}{\sqrt{r^2 - x^2}}$$

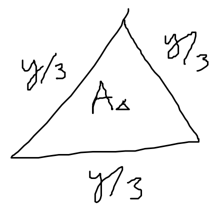
$$V'(c) = 0 \Rightarrow c = 0, \pm\sqrt{\tfrac{2}{3}}\,r$$

So $c = \sqrt{\tfrac{2}{3}}\,r$ is the relevant critical number

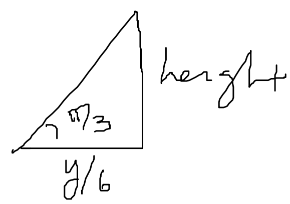
$$\boxed{V(c) = \frac{4}{9}\pi r^3 \sqrt{3}}$$

\# 31 A piece of wire 10 m long is cut into two pieces. One piece is bent into a square and the other into an equilateral triangle. How should the wire be cut so total area enclosed is maximum? min?





$$A_\triangle = \frac{1}{2}\,\text{base} \cdot \text{height}$$



$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{height}}{y/6}$$

$$\Rightarrow \text{height} = \frac{y}{6}\tan\left(\pi/3\right)$$

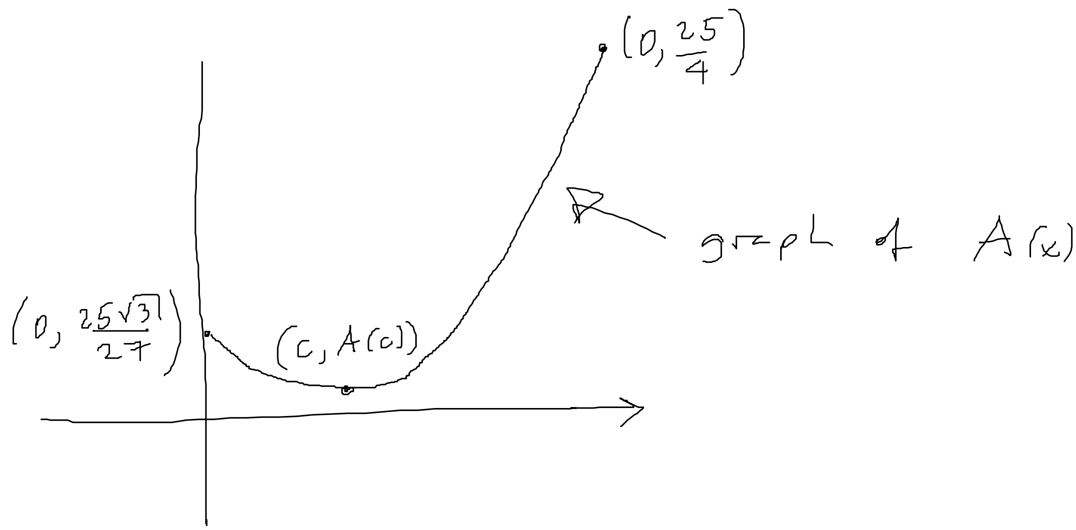
$$= \frac{y}{6}\,\frac{\sqrt{3}}{3}$$

$$= \frac{y}{6\sqrt{3}}$$

$$A_\square = \left(\frac{x}{4}\right)^2 \qquad A_\triangle = \frac{1}{2}\left(\frac{y}{3}\right)\left(\frac{y}{6\sqrt{3}}\right) = \frac{y^2}{36\cancel{2}\sqrt{3}}$$

$$A_\square + A_\triangle = \frac{x^2}{16} + \frac{(10-x)^2}{36\sqrt{3}}$$

total area $= \dfrac{x^2}{16} + \dfrac{(10-x)^2}{36\sqrt{3}}$



to maximize $A(x)$, take $x = 10 \implies y = 0 \implies$ all square & no triangle.

to minimize $A(x)$, $\quad A'(x) = \dfrac{x}{8} - \dfrac{\sqrt{3}(10-x)}{54}$

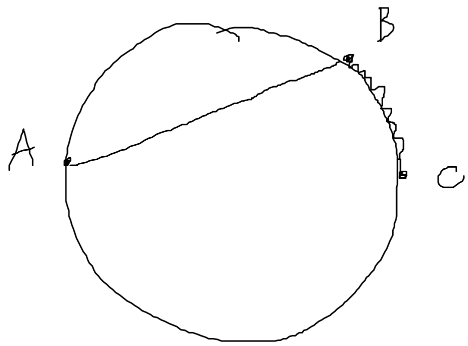
$$A'(c) = 0 \quad \text{if} \quad c = \frac{40\sqrt{3}}{27 + 4\sqrt{3}}$$

$$A(c) = \frac{25\sqrt{3}}{27 + 4\sqrt{3}} \cong 1.28$$

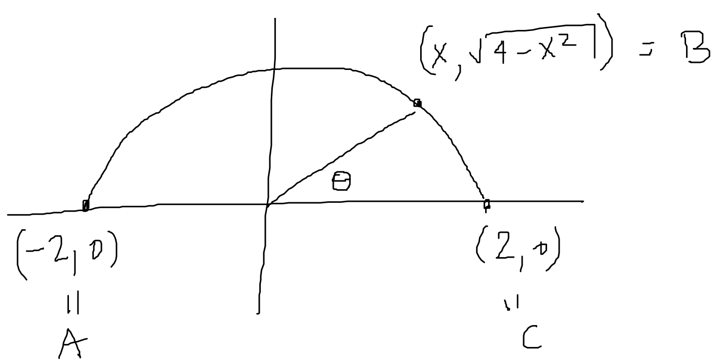
$$A(10) = \frac{25}{4} = 5.25$$

$$A(0) = \frac{25\sqrt{3}}{27} \cong 1.60$$

#42 A woman at point A on the shore of a circular lake with radius 2mi wants to arrive at point C diametrically opposite A on the other side of the lake in the shortest possible time. She can walk at 4mi/h and can row at 2mi/h. What should she do?



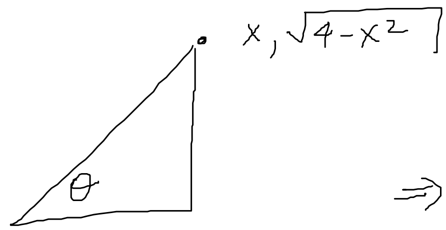
$$\text{time elapsed} = \frac{\text{dist}(A,B)}{2} + \frac{\text{dist}(B,C)}{4}$$



$$\text{dist}(A,B) = \sqrt{(x+2)^2 + (\sqrt{4-x^2} - 0)^2}$$

$$= \sqrt{(x+2)^2 + 4 - x^2} = 2\sqrt{x+2}$$

$\text{dist}(B, C) = 2\theta$



$$\Rightarrow \tan(\theta) = \frac{\sqrt{4-x^2}}{x}$$

$\Rightarrow$ if $x > 0$ then θ is in $[0, \pi/2)$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{4-x^2}}{x}\right)$$

if $x < 0$ then θ is in $(\pi/2, \pi]$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{4-x^2}}{x}\right) + \pi$$

$$\Rightarrow \text{dist}(B, C) = \begin{cases} 2\pi + 2\tan^{-1}\left(\frac{\sqrt{4-x^2}}{x}\right) & \text{if } x < 0 \\ 2\tan^{-1}\left(\frac{\sqrt{4-x^2}}{x}\right) & \text{if } x > 0 \end{cases}$$

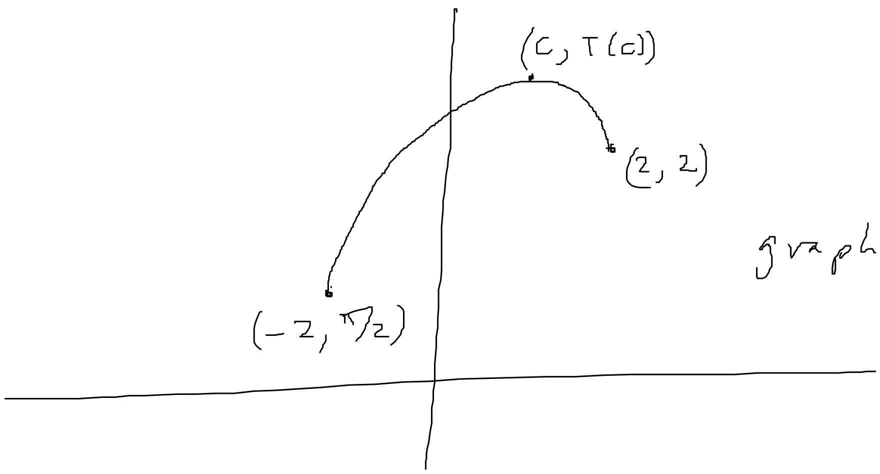
$$T(x) = \frac{\text{dist}(A, B)}{2} + \frac{\text{dist}(B, C)}{4}$$

$$= \begin{cases} \sqrt{x+2} + \frac{\pi}{2} + \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{4-x^2}}{x}\right) & \text{if } x < 0 \\ \sqrt{x+2} + \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{4-x^2}}{x}\right) & \text{if } x > 0 \end{cases}$$

know that if

$x = 2$ then she rowed a distance of 4 miles at $2 \, m/hr \Rightarrow$ took 2 hours.

$x = -2$ then she walked a distance of 2π miles at $4 \, mi/hr \Rightarrow$ took $\pi/2$ hours



graph of T

$$T'(x) = \frac{\sqrt{4-x^2} - \sqrt{x+2}}{2\sqrt{x+2}\,\sqrt{4-x^2}}$$

$T'(c) = 0 \Rightarrow c = 1$

$T(1) = \sqrt{3} + \frac{\pi}{6} \cong 2.26$

To minimize the time, she should walk. It'll take her $\frac{\pi}{2}$ hours $\cong$ 1 hr 34 min

To maximize the time, she should row to the point $(1, \sqrt{3})$, and then walk.
It'll take her $\sqrt{3} + \frac{\pi}{6}$ hours $\cong$ 2 hours 15 minutes.