Nov 22, 2004 Mat 135

§4.4 L'Hospital's Rule

You know what to do with

$$\lim_{x\to 1} \frac{2x^2+2x-3}{x^2-1}$$ since num $\to$ 1 and den $\to$ 0 the limit DNE.

$$\lim_{x\to 1} \frac{x^2-1}{2x^2+2x-3}$$ since num $\to$ 0 and den $\to$ 1 the limit equals 0.

$$\lim_{x\to 1} \frac{x^2-1}{x^2+2x-3}$$ num $\to$ 0 and den $\to$ 0 but you can factor!

$$= \lim_{x\to 1} \frac{(x+1)(x-1)}{(x+3)(x-1)} = \lim_{x\to 1} \frac{x+1}{x+3} = \frac{1}{2}$$

what about $\lim_{x\to 0} \frac{\sin(x)}{x}$? num $\to$ 0 and den $\to$ 0 but limit = 1.

How?

well, that one we can do w/ trig & geometry

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

if num $\to 0$ as $x \to a$
and den $\to 0$ as $x \to a$ there are three possibilities.

possibility 1: $f(x)$ goes to zero faster than $g(x)$ does. In this case the ratio goes to zero. (Example $f(x) = x^2$ and $g(x) = x^{3/2}$ as $x \to 0$.)

possibility 2: $f(x)$ goes to zero slower than $g(x)$ does. In this case, the ratio goes to infinity (Example: $f(x) = x^3$ and $g(x) = x^{5/2}$ as $x \to 0$.)

possibility 3: $f(x)$ goes to zero at the same rate as $g(x)$. In this case, the ratio goes to some nonzero number (Example: $f(x) = 2x^5$ and $g(x) = -3x^5$ as $x \to 0$.)

L'Hospital's rule is able to distinguish between these 3 possibilities.

Similarly, for

$\lim_{x \to a} \frac{f(x)}{g(x)}$ if num $\to \infty$ and denom $\to \infty$ then there are questions about how *fast* each goes.

L'Hospital's Rule Suppose f and g are differentiable and $g'(x) \neq 0$ near a.

Suppose that as $x \to a$

$$f(x) \to 0 \text{ and } g(x) \to 0$$

or that as $x \to a$

$$f(x) \to \infty \text{ and } g(x) \to \infty$$

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Note: similarly if $f(x) \to +\infty$ and $g(x) \to -\infty$

or if $f(x) \to -\infty$ and $g(x) \to +\infty$

or if $f(x) \to -\infty$ and $g(x) \to -\infty$.

ex: $\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1$

$\lim_{x \to 0} \frac{\cos(x)-1}{x} = \lim_{x \to 0} \frac{-\sin(x)}{1} = 0$

ex: $\lim_{x \to 0+} \frac{\ln x}{x}$

numer $\to -\infty$ den $\to 0$ so L'Hospital's rule doesn't apply. Limit DNE, and $\lim_{x \to 0+} \frac{\ln x}{x} = -\infty$

ex. $\lim_{x \to 1} \frac{x^9 - 1}{x^5 - 1} = \lim_{x \to 1} \frac{9x^8}{5x^4} = \lim_{x \to 1} \frac{9}{5} x^4 = 0$

ex $\lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{\csc \theta}$

recall $\csc(\theta) = \frac{1}{\sin(\theta)}$

numer $\to 0$ as $\theta \to \pi/2$

den $\to 1$ as $\theta \to \pi/2$

L'Hospital doesn't apply, $\lim_{\theta \to \pi/2} \frac{1 - \sin \theta}{\csc \theta} = 0$

ex: $\lim_{x\to 0} \dfrac{e^x - 1 - x - x^2/2}{x^3}$

$= \lim_{x\to 0} \dfrac{e^x - 1 - x}{3x^2}$

$= \lim_{x\to 0} \dfrac{e^x - 1}{6x}$

$= \lim_{x\to 0} \dfrac{e^x}{6} = \dfrac{1}{6}$

Why does L'Hospital's rule work?
We haven't learnt about Taylor polynomials yet, but when we do, we'll find that for x close to a,

$$f(x) \cong f(a) + f'(a)(x-a)$$

$$g(x) \cong g(a) + g'(a)(x-a)$$

and so

$$\frac{f(x)}{g(x)} \cong \frac{f(a) + f'(a)(x-a)}{g(a) + g'(a)(x-a)}$$

$$= \frac{f'(a)}{g'(a)}$$

Note: the Taylor polynomials argument only works for a finite and for f and g differentiable, while l'Hospital's rule is very flexible, also working for $a = \pm\infty$.

ex:

$$\lim_{x\to\infty} \frac{\ln(\ln(x))}{x}$$ numer $\to \infty$, den $\to \infty$

$$= \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{1} = \lim_{x\to\infty} \frac{1}{x\ln x} = 0$$

ex:

$$\lim_{x\to\infty} \frac{e^x}{x^3}$$ numer $\to \infty$ \$ den $\to \infty$

$$= \lim_{x\to\infty} \frac{e^x}{3x^2}$$ numer $\to \infty$ \$ den $\to \infty$

$$= \lim_{x\to\infty} \frac{e^x}{6x}$$ numer $\to \infty$ \$ den. $\to \infty$

$$= \lim_{x\to\infty} \frac{e^x}{6}$$

limit DNE

$$\lim_{x\to\infty} \frac{e^x}{x^3} = \infty$$

L'Hospital's rule also helps with

$$\lim_{x \to a} f(x)\, g(x)$$

when $f(x) \to 0$ and $g(x) \to \infty$ (or $-\infty$).

Why?

$$\lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} \frac{f(x)}{1/g(x)} \quad \text{and}$$

numer $\to 0$ & den $\to 0$ so L'Hôpital's rule applies!

ex: $\displaystyle\lim_{x \to \infty} x^3 e^{-x^2}$ $\qquad$ $x^3 \to \infty$ as $x \to \infty$

$e^{-x^2} \to 0$ as $x \to \infty$

$$= \lim_{x \to \infty} \frac{x^3}{1/e^{-x^2}} = \lim_{x \to \infty} \frac{x^3}{e^{x^2}}$$

$$= \lim_{x \to \infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x \to \infty} \frac{3x}{2e^{x^2}}$$

num $\to \infty$ & den $\to \infty$

$$= \lim_{x \to \infty} \frac{3}{4xe^{x^2}}$$

num $\to 3$ & den $\to \infty$

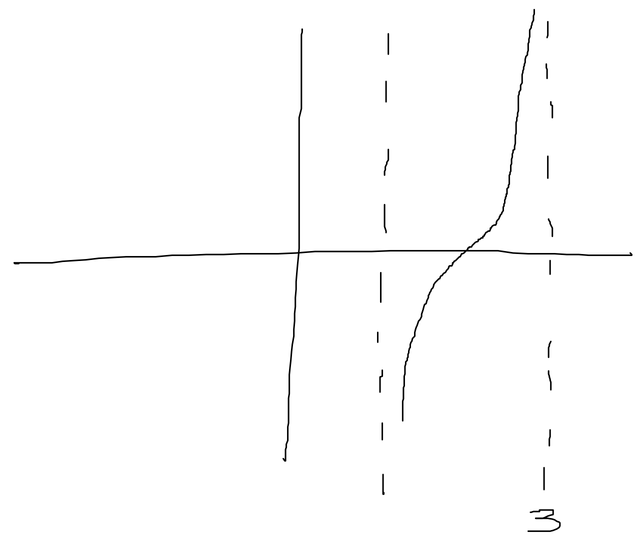
$$= 0$$

ex: $\displaystyle\lim_{x\to 1^+} \ln(x)\tan\left(\frac{\pi}{2}x\right)$

$\ln(x) \to 0$ as $x \to 1^+$

$\tan\left(\frac{\pi}{2}x\right) \to -\infty$ as $x \to 1^+$

want to use L'Hospital's rule.



graph of $\tan\left(\frac{\pi}{2}x\right)$

option 1:

$$\lim_{x\to 1^+} \frac{\ln(x)}{\frac{1}{\tan\left(\frac{\pi}{2}x\right)}}$$

option 2:

$$\lim_{x\to 1^+} \frac{\tan\left(\frac{\pi}{2}x\right)}{1/\ln(x)}$$

experience tells us to go for option 1 since $\ln(x)$ has a derivative that's better-looking than $1/\ln(x)$.

$$\lim_{x \to 1^+} \ln|x| \tan\left(\frac{\pi}{2}x\right)$$

$$= \lim_{x \to 1^+} \frac{\ln|x|}{1/\tan(\pi/2\, x)}$$

$$= \lim_{x \to 1^+} \frac{1/x}{-\frac{\pi}{2} \frac{1}{\sin^2(\pi/2\, x)}}$$

$$\frac{1}{x} \to 1$$

$$\frac{1}{\sin^2(\frac{\pi}{2}x)} \to 1$$

$$= -\frac{2}{\pi}$$

Finally, L'Hospital's rule is helpful for

$$\lim_{x \to a} f(x)^{g(x)}$$

where $f \to 0$ and $g \to 0$ (type 0^0)

or $f \to \infty$ and $g(x) \to 0$ (type ∞^0)

or $f(x) \to 1$ and $g(x) \to \infty$ (type 1^∞)

ex: $\lim_{x \to 0} (1-2x)^{\frac{1}{x}}$ $\qquad 1-2x \to 1$

$\frac{1}{x} \to 0$

let $y = (1-2x)^{\frac{1}{x}}$

then $\ln(y) = \frac{1}{x} \ln(1-2x)$

and $\lim_{x \to 0} \ln(y) = \lim_{x \to 0} \frac{\ln(1-2x)}{x}$ $\qquad$ num $\to 0$ \$ den $\to 0$

$$= \lim_{x \to 0} \frac{\frac{-2}{1-2x}}{1}$$

$$= \lim_{x \to 0} \frac{-2}{1-2x} = -2$$

So as $x \to 0$ $\quad \ln\left[(1-2x)^{\frac{1}{x}}\right] \to -2$

since exp is a continuous function,

as $x \to 0$ $\quad \exp\left(\ln\left[(1-2x)^{\frac{1}{x}}\right]\right) \to \exp(-2)$

and so $\lim_{x \to 0} (1-2x)^{1/x} = e^{-2}$

ex: $\lim_{x \to 0+} (\cos(x))^{\frac{1}{x^2}}$ as $x \to 0+$ $\cos(x) \to 1$

$\frac{1}{x^2} \to \infty$

$$\lim_{x \to 0+} \ln\left[\cos(x)^{\frac{1}{x^2}}\right] = \lim_{x \to 0+} \frac{1}{x^2} \ln(\cos(x))$$

$$= \lim_{x \to 0+} \frac{\ln(\cos(x))}{x^2}$$

$$= \lim_{x \to 0+} \frac{\frac{-\sin(x)}{\cos(x)}}{2x}$$ (used L'Hospital's rule!)

$$= \lim_{x \to 0+} \frac{-\sec^2(x)}{2}$$

$$= -1/2$$

since exp is a continuous function, since

$\ln\left[\cos(x)^{1/x^2}\right] \to -1/2$ as $x \to 0+$ then

$\exp\left(\ln\left(\cos(x)^{1/x^2}\right)\right) \to \exp(-1/2)$ as $x \to 0+$

And so, $\lim_{x \to 0+} (\cos(x))^{1/x^2} = e^{-1/2}$