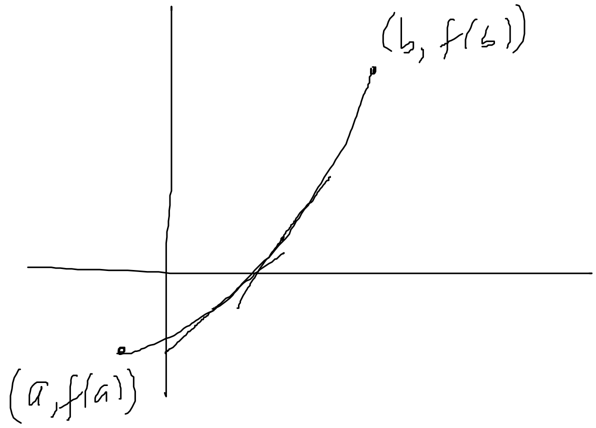
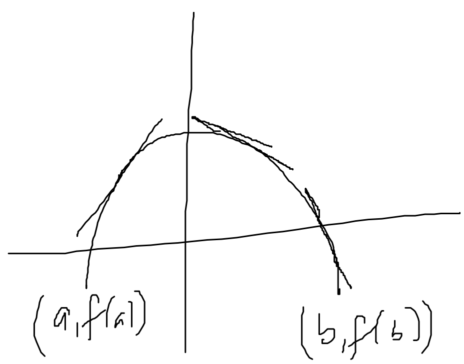
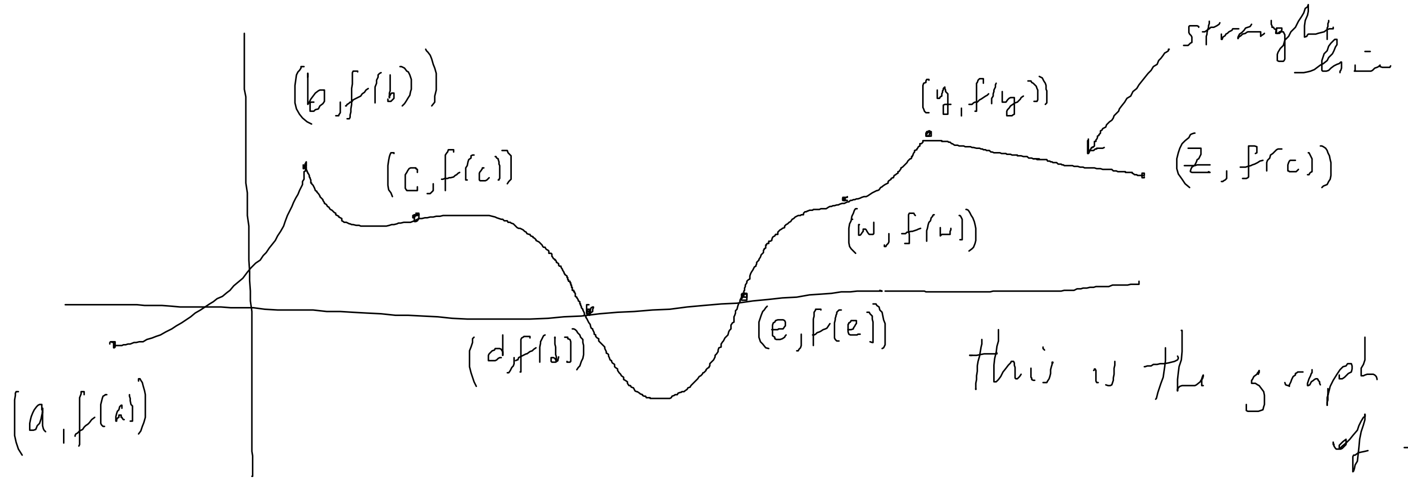
Mat 135 Nov 19, 2004.



the graph of f is concave up on (a, b). why? because the graph of f lies <u>above</u> all the tangent lines to the graph



the graph of f is concave down on (a, b). why? because the graph of f lies <u>below</u> all the tangent lines to the graph.



this is the graph of f.

Where is the graph of f concave up? concave down?

the graph of f is concave up on (a, b)
the graph of f is concave up on (b, c)
the graph of f is concave down on (c, d)
the graph of f is concave up on (d, e)
the graph of f is concave down on (e, w)
the graph of f is concave up on (w, y)

the graph is both concave up <u>and</u> concave down on (y, z).

<u>definition</u>: a point p on a curve $y = f(x)$ is an <u>inflection point</u> if f is continuous at p and if the curve changes from concave up to concave down or from concave down to concave up at p.

For example, in the previous graph,
c, d, e, and w are inflection points

Theorem (concavity test)

1) if $f'' > 0$ on (a,b) then the graph of f is concave up on (a,b).

2) If $f'' < 0$ on (a,b) then the graph of f is concave down on (a,b).

Theorem: (second derivative test) Suppose f'' is continuous near c.

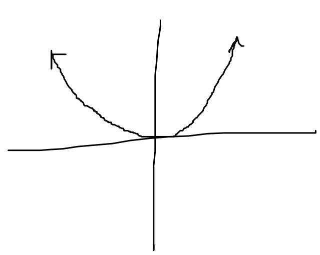
1) if $f'(c) = 0$ and $f''(c) > 0$ then c is a local minimum.

2) if $f'(c) = 0$ and $f''(c) < 0$ then c is a local maximum.

ex: $f(x) = x^2$

then $f'(x) = 2x$ $\qquad$ $f'(c) = 0$ at $c = 0$

$f''(x) = 2$ $\qquad$ $f''(c) > 0$



$\Rightarrow$ c a local min

#22 Find the local maxima and local minima using both the first and second derivative tests.

$$f(x) = \frac{x}{x^2+4}$$

$$f'(x) = \frac{4-x^2}{(x^2+4)^2}$$

$\Rightarrow$ $f'<0$ | $f'>0$ | $f'<0$

-2 $\quad$ 2

the critical points are -2 and 2.

first derivative test:

f' changes from neg to pos at $-2 \Rightarrow -2$ local min

f' changes from pos to neg at $2 \Rightarrow 2$ local max

$$f''(x) = \frac{2x(x^2-12)}{(x^2+4)^3}$$

$\sqrt{12} = 2\sqrt{3}$

$\Rightarrow$ $f''<0$ | $f''>0$ | $f''<0$ | $f''>0$

$-2\sqrt{3}$ $\quad$ 0 $\quad$ $2\sqrt{3}$

second derivative test

$f''(-2) > 0 \Rightarrow -2$ local min

$f''(2) < 0 \Rightarrow -2$ local max.

Now to plot f!

observation 1: $\lim_{x \to \infty} f(x) = 0$

$\lim_{x \to -\infty} f(x) = 0$

observation 2: $f(x) = 0$ only at $x = 0$

f is an odd function.

<u>facts:</u> $-2\sqrt{3}, 0, 2\sqrt{3}$ are inflection points

-2 local min

2 local max

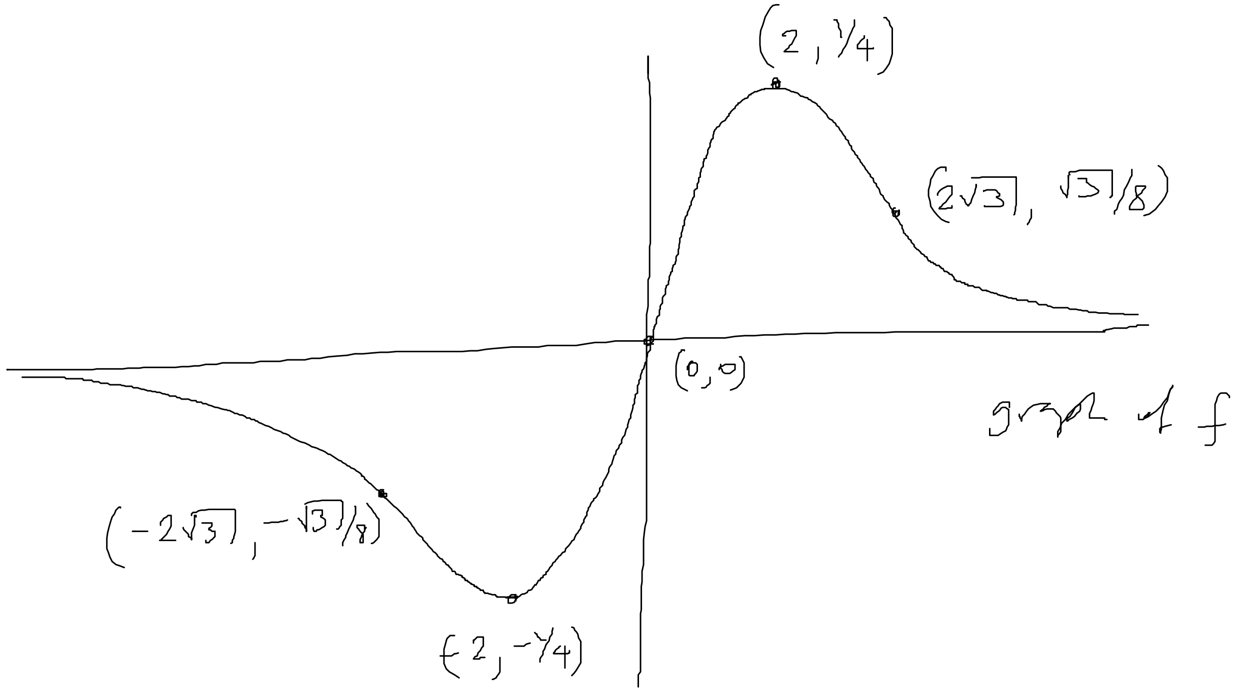
$f(-2\sqrt{3}) = -\sqrt{3}/8 \cong -.22$

$f(-2) = -1/4$

$f(0) = 0$

$f(2) = 1/4$

$f(2\sqrt{3}) = \sqrt{3}/8 \cong .22$



ex: 41 $\quad f(x) = x^{1/3}(x+4)$

analyze & graph.

$$f'(x) = \frac{4(x+1)}{3x^{2/3}}$$

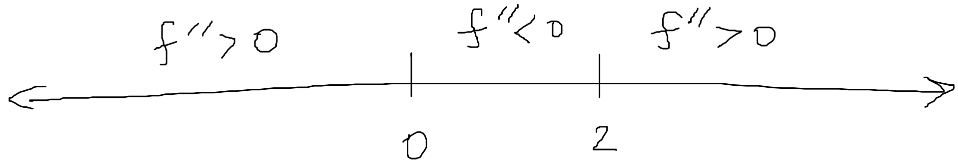
$$f''(x) = \frac{4(x-2)}{9x^{5/3}}$$

critical points of f : -1 and 0

$$\begin{array}{ccccc} f'<0 & & f'>0 & & f'>0 \\ \longleftarrow & | & \text{———} & | & \longrightarrow \\ & -1 & & 0 & \end{array}$$

first derivative test $\Rightarrow$ -1 local min

0 neither local min nor l. max



from the second derivative,

0 is an inflection point, as is 2.

<u>observations</u>

$\lim_{x\to\infty} f(x) = \infty \qquad f(x) \sim x^{4/3}$

$\lim_{x\to-\infty} f(x) = \infty \qquad f(x) \sim x^{4/3}$

$f(x) = 0$ at $x = 0$ and $x = 4$

calculus facts: local min at -1

vertical tangent at 0

with $f' \to \infty$ as $x \to 0-$

$f' \to \infty$ as $x \to 0+$

f increasing on $(-1, 0)$

decreasing on $(-\infty, -1)$ and $(0, \infty)$

f concave up on $(-\infty, 0)$ and $(2, \infty)$

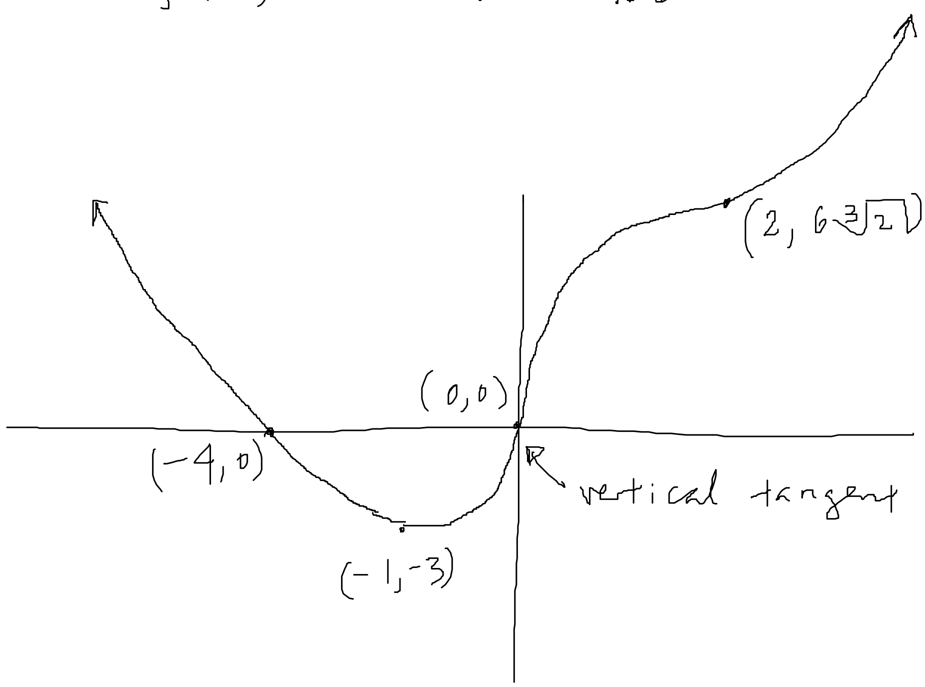
concave down on $(0, 2)$

$f(-4) = 0$

$f(-1) = -3$

$f(0) = 0$

$f(2) = 2^{1/3} \cdot 6 \cong 7.6$

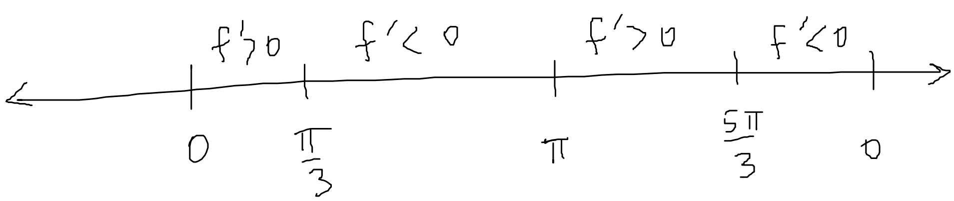


ex #43 $\quad f(x) = 2\cos x - \cos(2x)$ on $[0, 2\pi]$

$f'(x) = -2\sin x + 2\sin(2x)$

$f''(x) = -2\cos(x) + 4\cos(2x)$

$f'(x) = 0$ at $x = 0, \pi, \pi/3, 5\pi/3, 2\pi$



$\Rightarrow$ by first derivative test

$\frac{\pi}{3}$ local max

π local min

$\frac{5\pi}{3}$ local max,

(0 and 2π can't be local extrema since they're endpoints)

$$f''(x) = -2\cos x + 4\cos(2x)$$

$$= 8\cos^2(x) - 2\cos x - 4$$

$$= 8y^2 - 2y - 4 \quad \text{where } y = \cos(x)$$

$$= 8\left(y - \frac{1+\sqrt{33}}{8}\right)\left(y - \frac{1-\sqrt{33}}{8}\right)$$

$$= 8\left(\cos(x) - \frac{1+\sqrt{33}}{8}\right)\left(\cos(x) - \frac{1-\sqrt{33}}{8}\right)$$

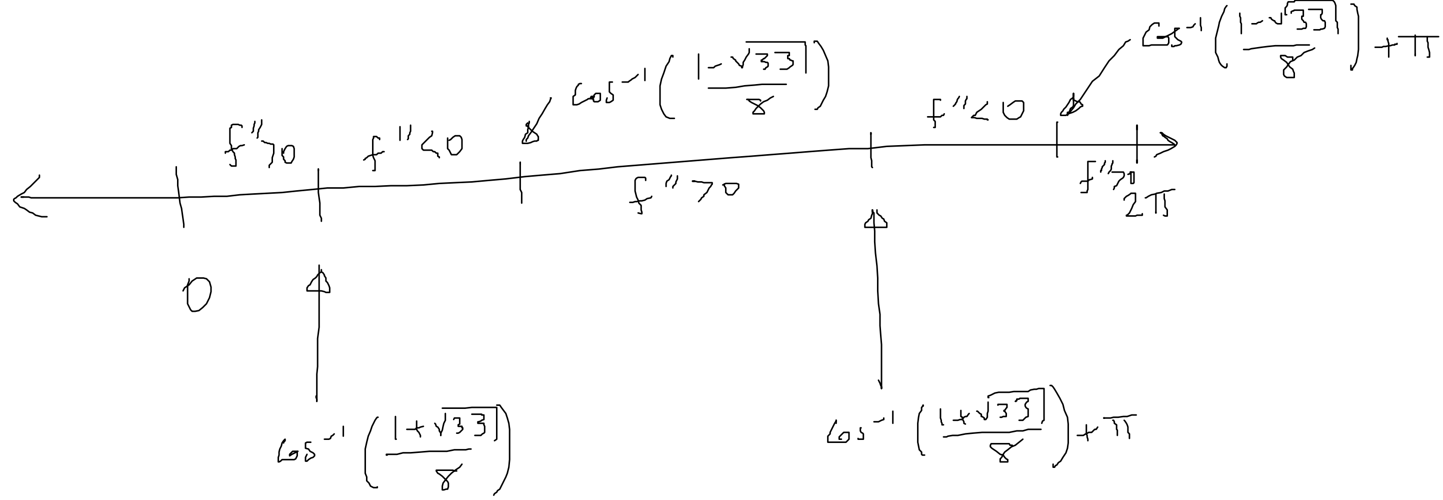
find x using $\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right)$ and $\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right)$

$f'' = 0$ at $\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right) \cong .57$

$\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right) \cong 2.21$

$$\cos^{-1}\left(\frac{1+\sqrt{33}}{8}\right)+\pi \cong 3.71$$

$$\cos^{-1}\left(\frac{1-\sqrt{33}}{8}\right)+\pi \cong 5.35$$



Now we know the function has four inflection points, two local maxima, and one local minimum

$f(0) = 1$

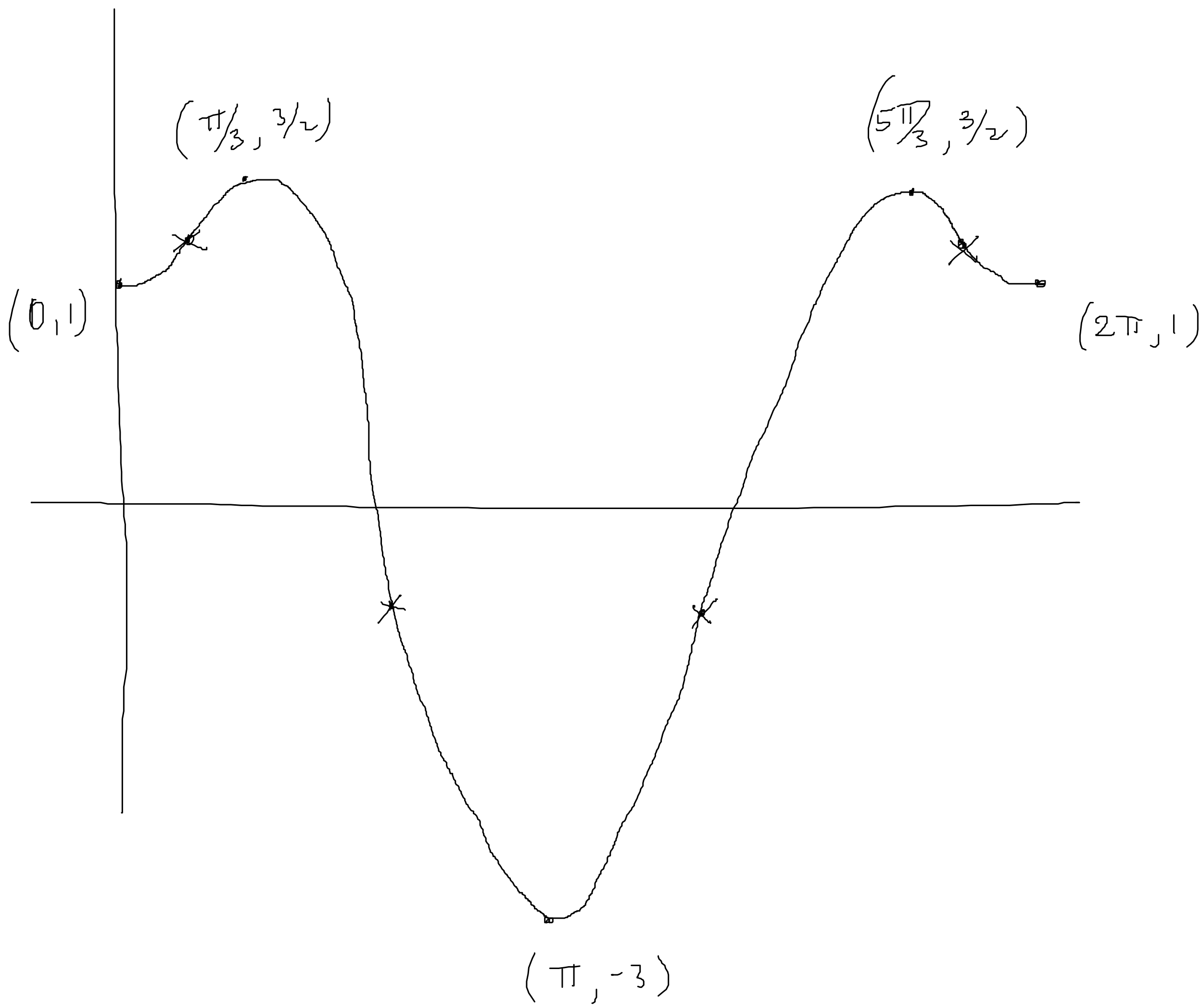
$f\left(\frac{\pi}{3}\right) = 3/2$

$f(\pi) = -3$

$f\left(\frac{5\pi}{3}\right) = 3/2$

$f(2\pi) = 1$

graph of f:



the four inflection points are marked with x's.