Mat 135, Nov 17 2004

Recall what it means for a function to be increasing:

<u>definition</u>: f is <u>increasing</u> on $[a,b]$ if for any x and y in $[a,b]$, if $x<y$ then $f(x)<f(y)$. Similarly, f is <u>decreasing</u> on $[a,b]$ if for any x and y in $[a,b]$ if $x<y$ then $f(x)>f(y)$.

<u>For example</u>: $f(x)=x^2$ is decreasing on $(-\infty,0]$ and increasing on $[0,\infty)$.

1) Why is f decreasing on $(-\infty,0]$? Let x and y be in $(-\infty,0]$. If $x<y$ then $x-y<0$. But $x+y<0$ since x and y are in $(-\infty,0]$ so $(x-y)(x+y)>0$. That is, $x^2-y^2>0$ $\Rightarrow$ $x^2>y^2$ and this shows $f(x)>f(y)$.

2) Why is f increasing on $[0,\infty]$? Let x and y be in $[0,\infty)$. If $x<y$ then $x-y<0$. But $x+y>0$ since x and y are in $[0,\infty]$. So $(x-y)(x+y)<0 \Rightarrow x^2-y^2<0$ $\Rightarrow x^2<y^2 \Rightarrow f(x)<f(y)$. This shows f is increasing on $[0,\infty)$.

That's a lot of work! Is there an easier way to tell if a function is increasing on an interval?

Yes! If the function is differentiable, then we can use the mean value theorem.

Let $x<y$. If f is continuous on $[x,y]$ and is differentiable on (x,y) then the MVT implies there's at least one number c in (x,y) such that

$$f'(c) = \frac{f(y)-f(x)}{y-x}.$$

Therefore $f(y) - f(x) = f'(c)\underbrace{(y-x)}_{\text{definitely positive.}}$

If $f'(c) > 0$ then $f(y) - f(x) > 0$

if $f'(c) < 0$ then $f(y) - f(x) < 0$.

<u>Theorem</u>: (The increasing/decreasing test)

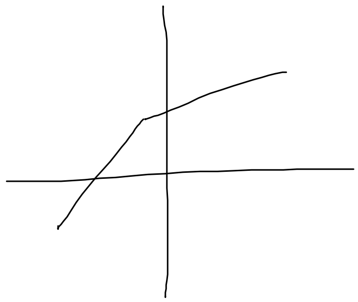
1) if $f' > 0$ on (a,b) then f is increasing on (a,b).

2) if $f' < 0$ on (a,b) then f is decreasing on (a,b).

<u>NOTE</u> 1: f increasing on (a,b) does not imply $f' > 0$ on (a,b).

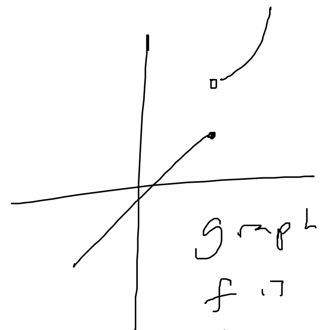
why? $f(x) = x^3$ is increasing on $(-1,1)$ but $f'(x) = 3x^2 \not> 0$ on $(-1,1)$.

NOTE 2! you don't need f to be differentiable in order for f to be increasing. In fact you don't even need f to be continuous.



graph of f.

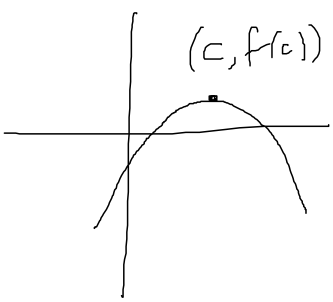
f is increasing



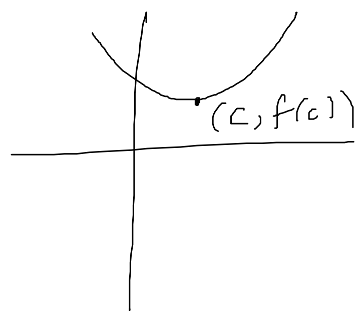
graph of f

f is increasing.

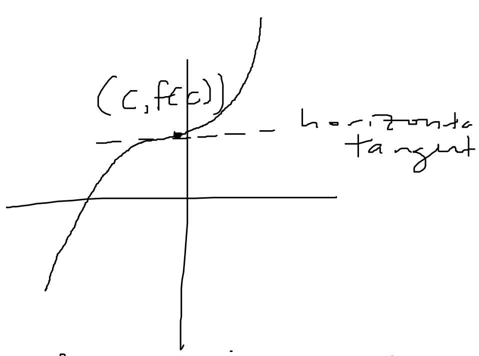
Now that we know how derivatives relate to increasing/decreasing we can say something about local maxima & local minima.



f increasing to left of c
f decreasing to right of c



f decreasing to left of c
f increasing to right of c



f increasing to left
f increasing to right

Theorem: (the First Derivative test)

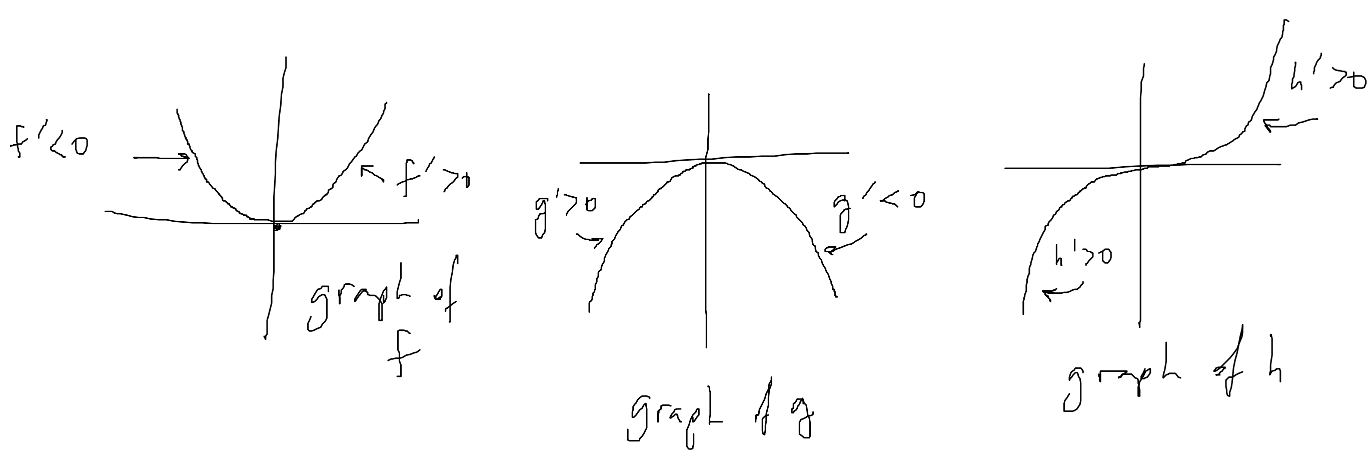
Suppose c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c then f has a local max at c

(b) If f' changes from negative to positive at c then f has a local min at c

(c) If f' is positive to both sides of c or is negative to both sides of c then c is neither a local max nor a local min.

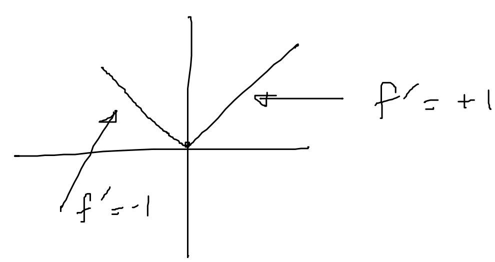
Note: to memorize the first derivative test all you need to do is understand $f(x) = x^2$, $g(x) = -x^2$ and $h(x) = x^3$



Note: The first derivative test can't tell you anything about absolute maxima or absolute minima. Why? To have an absolute max. at c, $f(c)$ has to be greater than or equal to $f(x)$ for all x in the domain D. The first derivative test only looks at the behavior of f near to x. So f could be doing all sorts of wild things on other parts of the domain.

Note: You don't need f to be differentiable at c to apply the first derivative test:

$f(x) = |x|$ on $[-1, 1]$.

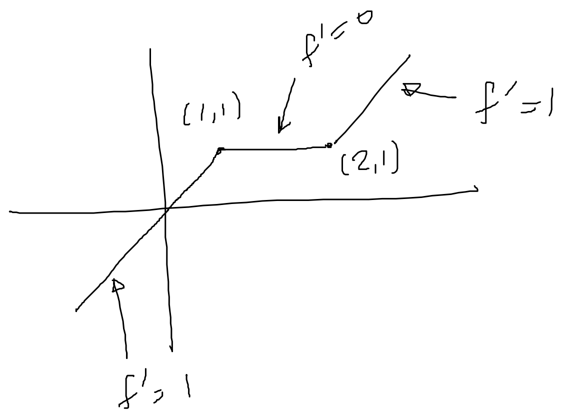


f' changes from negative to positive at c. $\Rightarrow$ c is a local min

Note:

Even if you can't apply the first derivative test, c might still be a local max or local min. For example

$$f(x) = \begin{cases} x & \text{if} \quad x \le 1 \\ 1 & \text{if} \quad 1 < x \le 2 \\ x-1 & \text{if} \quad x > 2 \end{cases}$$

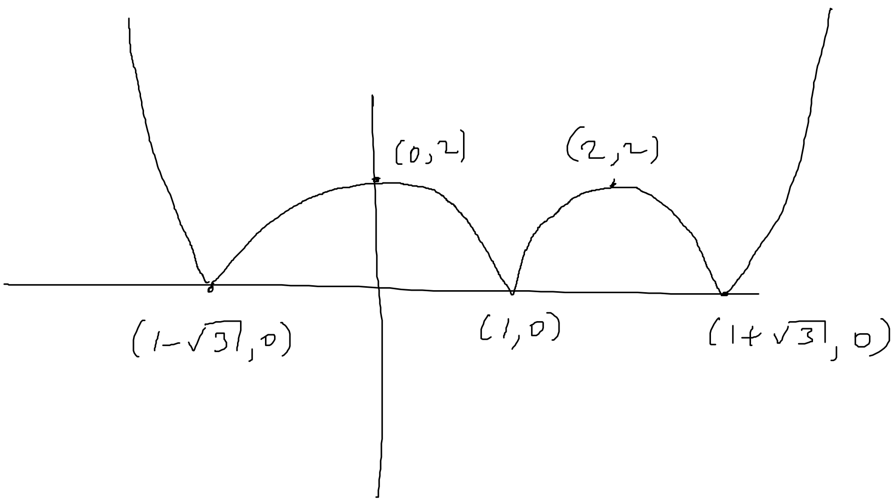


f has a local max at 1 and has a local min at 2.

f' changes from positive to zero at 1 ⇒ the first derivative test doesn't apply

f' changes from zero to positive at 2 ⇒ the first derivative test doesn't apply ex: $f(x) = |x^3 - 3x^2 + 2|$.

From Friday Nov 12

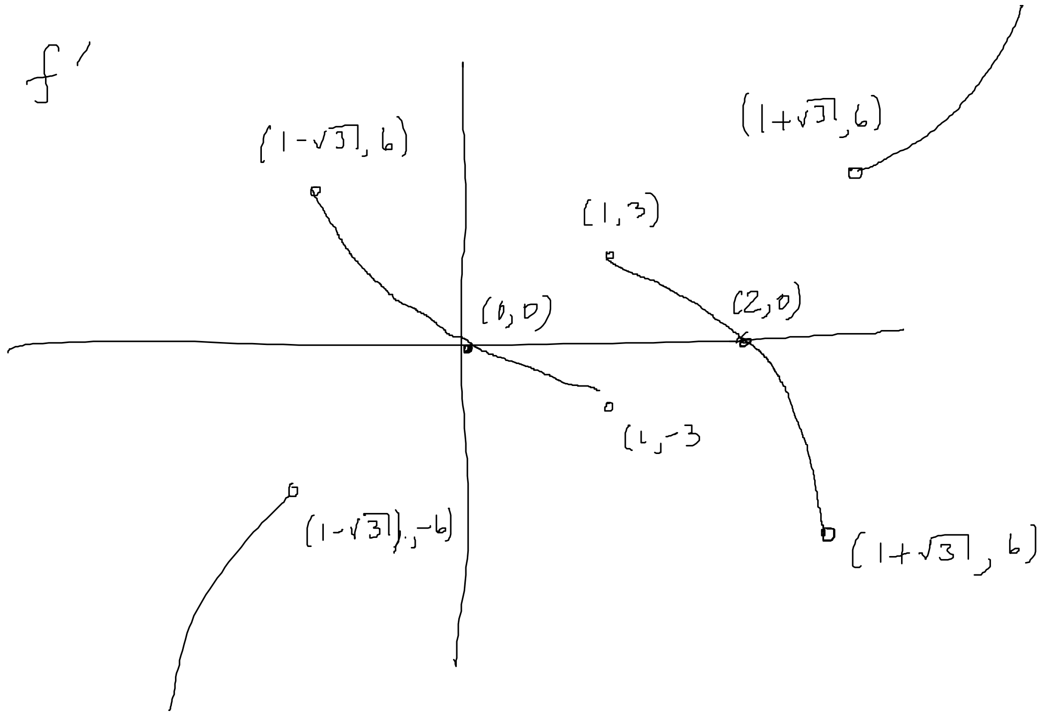


graph of f

$$f(x) = \begin{cases} -(x^3-3x^2+2) & x \le 1-\sqrt{3} \\ x^3-3x^2+2 & 1-\sqrt{3} < x \le 1 \\ -(x^3-3x^2+2) & 1 < x \le 1+\sqrt{3} \\ x^3-3x^2+2 & 1+\sqrt{3} < x \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -(3x^2-6x) & x < 1-\sqrt{3} \\ 3x^2-6x & 1-\sqrt{3} < x < 1 \\ -(3x^2-6x) & 1 < x < 1+\sqrt{3} \\ 3x^2-6x & 1+\sqrt{3} < x \end{cases}$$

graph of f'



f' changes from neg to pos at $1-\sqrt{3}$

$\Rightarrow$ $1-\sqrt{3}$ is local min

f' changes from pos to neg at 0

$\Rightarrow$ 0 is local max

f' changes from neg to pos at 1

$\Rightarrow$ 1 is local min

f' changes from pos to neg at 2

$\Rightarrow$ 2 is local max

f' changes from neg to pos at $1+\sqrt{3}$

$\Rightarrow$ $1+\sqrt{3}$ is local min