Ma 135, Nov 15 2004

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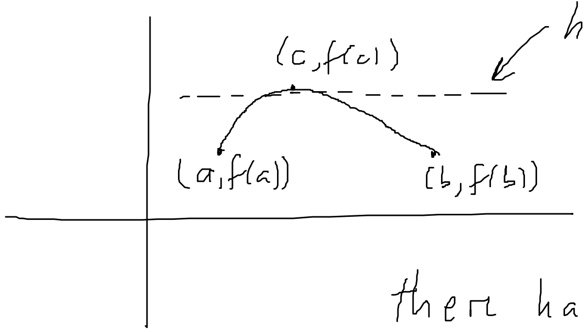
§4.2 Mean Value Theorem

Rolle's Theorem: Let f be a function that satisfies the following three hypotheses:

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)
3) $f(a) = f(b)$

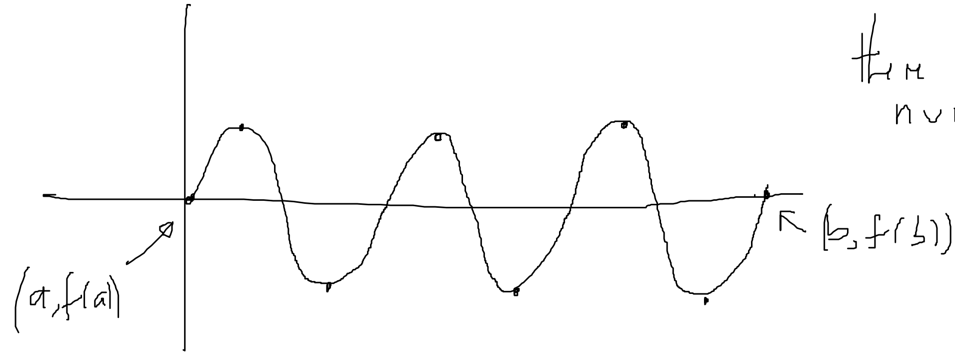
Then there is some c in (a,b) such that $f'(c) = 0$.

That is



there has to be at least one such c. Of course, there could be more than one

consider $\sin(x)$ on $[0, 6\pi]$

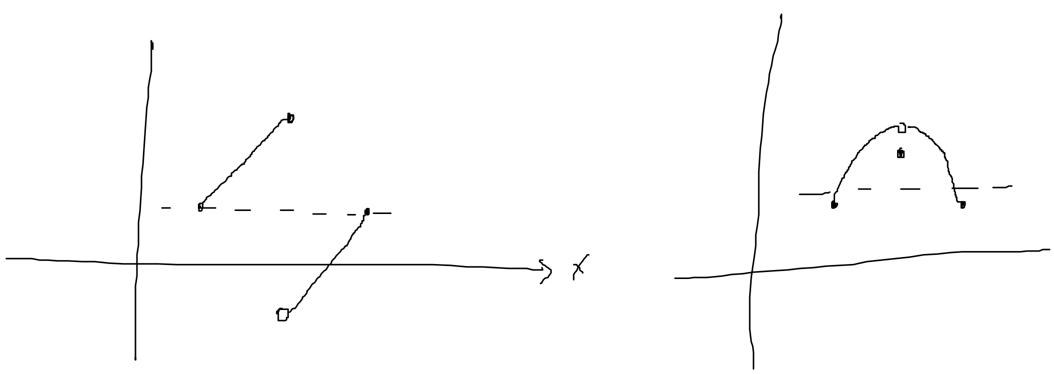


there are 6 different numbers in $(0, 6\pi)$ at which $f'(c) = 0$

Q: Do we really need to know all three things (f cont. on $[a,b]$, f differentiable on (a,b), $f(a)=f(c)$) in order to be guaranteed that $f'(c)=0$ at some point in (a,b)?

A: Yes! You need all three conditions.

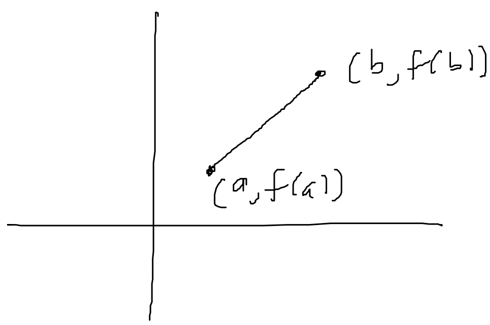
(1) Why do you need continuity? Here are two examples of functions on $[a,b]$ where $f(a)=f(b)$ but there's no point in (a,b) at which $f'(c)=0$.



(2) Why do you need differentiability on (a,b)? Here's a function that's continuous on $[a,b]$ and $f(a)=f(b)$ but there's no c in (a,b) such that $f'(c)=0$

continuous on $[a,b]$
and $f(a)=f(b)$, but
there's no c in (a,b) where
$f'(c)=0$

(3) Why do you need $f(a)=f(b)$? Here's a function that's continuous on $[a,b]$, differentiable on (a,b) but there's no c in (a,b) such that $f'(c)=0$



ex #4 Let $f(x) = x^3 - 3x^2 + 2x + 5$ on $[0,2]$

What can Rolle's theorem tell us?

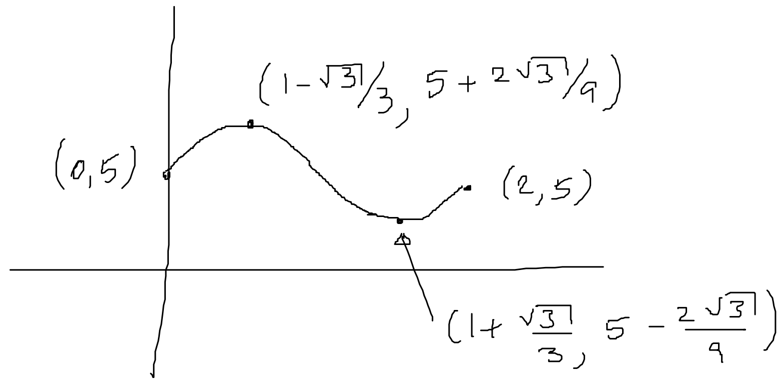
Can we apply Rolle's theorem?

continuous on $[0,2]$? Yes!
differentiable on $(0,2)$? Yes!
$f(0)=f(2)$? Yes!

$$f'(x) = 3x^2 - 6x + 2$$

$$= 3\left(x - \left(1 + \sqrt{3}/3\right)\right)\left(x - \left(1 - \sqrt{3}/3\right)\right)$$

graph of f



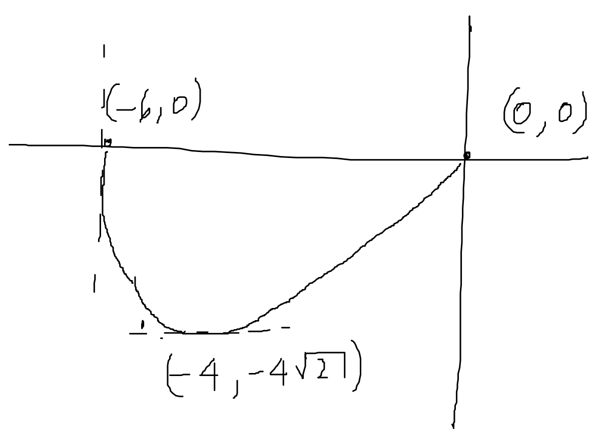
ex # 4 $\quad f(x) = x\sqrt{x+6}$ on $[-6,0]$.

continuous on $[-6,0]$? Yes!

differentiable on $(-6,0)$? yes!

$f(-6) = f(0)$? Yes

$$f'(x) = \frac{3(x+4)}{2\sqrt{x+6}}$$



$f'(c) = 0$ at $c = -4$ which is in $(-6,0)$

The Mean Value theorem is very similar to Rolle's theorem:

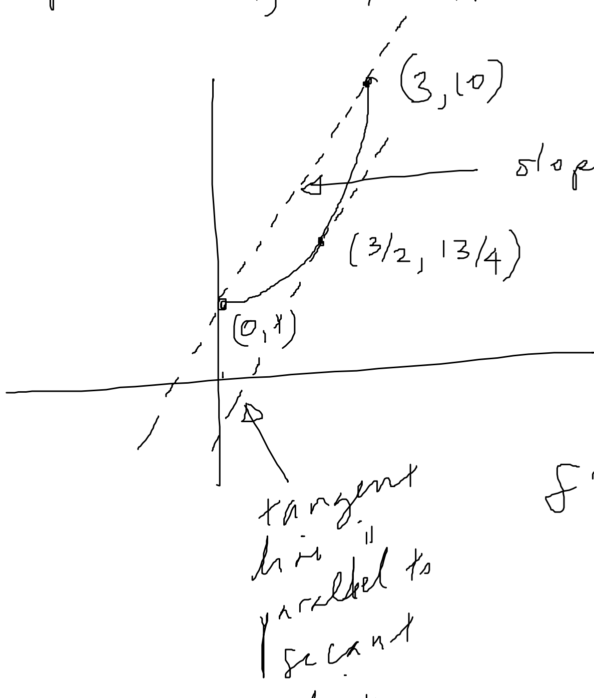
<u>MVT</u>: Let f be a function that satisfies the two hypotheses:

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)

Then there is a point c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

<u>example</u>: $f(x) = x^2+1$ on $[0,3]$



$\text{slope} = \frac{f(b)-f(a)}{b-a} = \frac{10-1}{3-0} = 3$

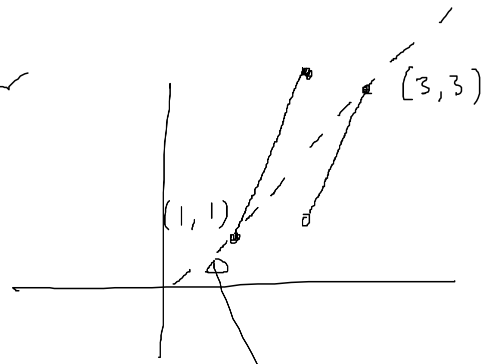
$f'(x) = 2x$

$f'(c) = 2c = 3$ at $c = 3/2$

So we've found c in $(0,3)$ such that $f'(c) = \frac{f(3)-f(0)}{3-0}$

Q: Do we really need f to be continuous on $[a,b]$ for the MVT to hold?

A: Yes consider

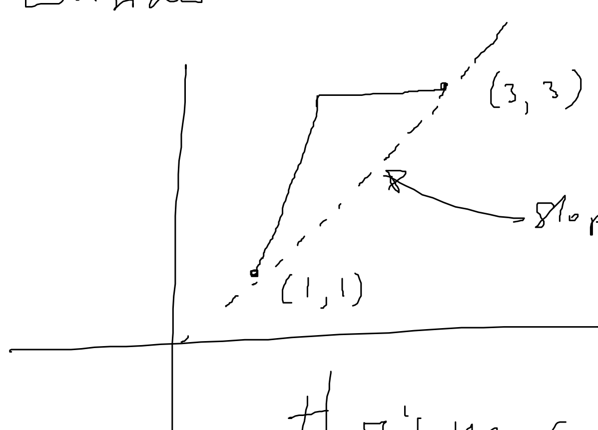


slope $= \dfrac{f(3)-f(1)}{3-1} = \dfrac{3-1}{3-1} = 1$

but there's no c in $(1,3)$ with $f'(c) = 1$.

Q: Do we really need f to be differentiable on (a,b) for the MVT to hold?

A: Yes. Consider



there's no c in $(1,3)$ with $f'(c) = 1$.

ex 13 let $f(x) = e^{-2x}$ on $[0, 2]$

apply MVT to find c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

ans: $\frac{f(2)-f(0)}{2-0} = \frac{e^{-4} - 1}{2}$

$f'(x) = -2e^{-2x}$

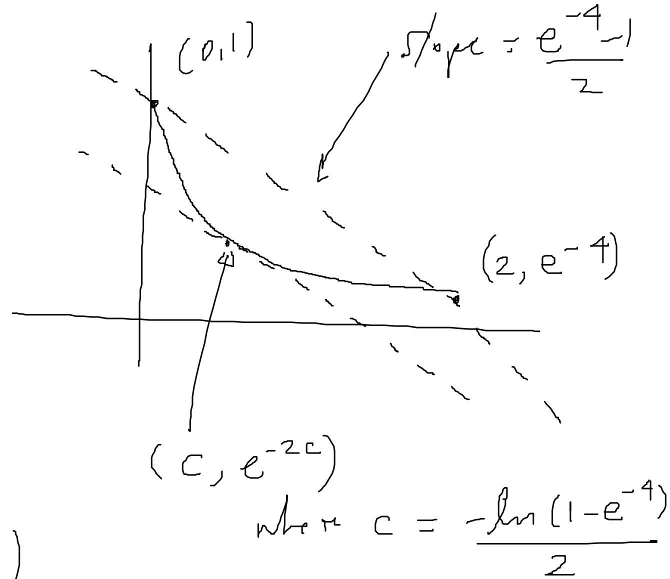
$f'(c) \overset{?}{=} \frac{e^{-4} - 1}{2}$

$= -2e^{-2c}$

$\Rightarrow e^{-2c} = 1 - e^{-4}$

$\Rightarrow -2c = \ln(1 - e^{-4})$

$\Rightarrow c = -\frac{1}{2} \ln(1 - e^{-4}) \cong 0.702$



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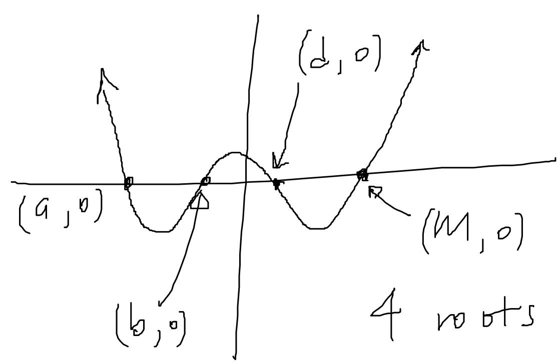
Show that the equation

$$x^4 + 4x + c = 0$$

has at most two real roots.

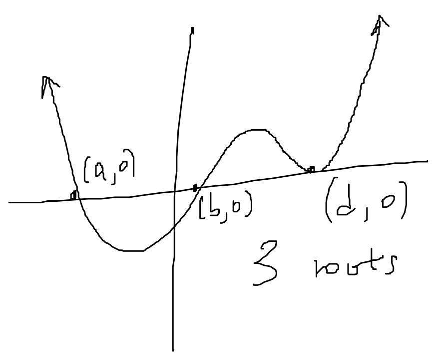
A: What would it mean if $f(x) = x^4 + 4x + c$ equalled zero at more than two places?

Then the graph would look like



4 roots $\Rightarrow f' = 0$ at 3 places by applying the MVT. to $[a, b]$ and to $[b, d]$ and to $[d, M]$

or



3 roots $\Rightarrow f' = 0$ at 3 places. two of them guaranteed by apply the MVT to $[a, b]$ and to $[b, d]$

idea: let's look at f' and see how often it can equal zero.

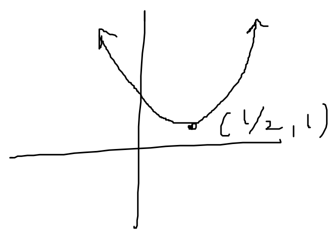
$$f'(x) = 4x^3 + 4 = 4(x^3+1)$$

$$4(x+1)(x^2-x+1)$$

$(x+1)$ equals zero at -1

can (x^2-x+1) ever equal zero? No!

It's a parabola ↑↑ with minimum at $1/2$



so f' can equal zero at only one point.

This means that $f(x)=0$ has at most two real roots

ex 24. Suppose $3 \le f'(x) \le 5$ for all values of x. Show that $18 \le f(8) - f(2) \le 30$.

Since $f'(x)$ exists for all x in $(-\infty, \infty)$, we know* f is continuous on $(-\infty, \infty)$ and so we can apply the MVT to $[2,8]$.

know $\dfrac{f(8)-f(2)}{8-2} = f'(c)$ for some c in $(2,8)$.

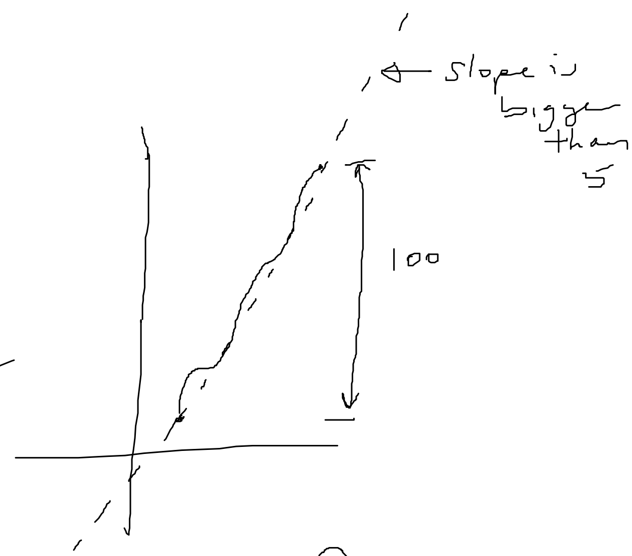
by the MVT.

$\Rightarrow \quad 3 \le \dfrac{f(8)-f(2)}{8-2} \le 5$ since $3 \le f'(c) \le 5$

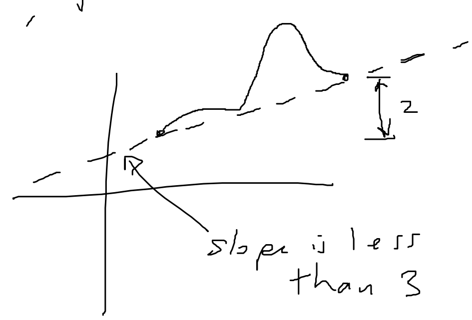
$\Rightarrow \quad 3 \le \dfrac{f(8)-f(2)}{6} \le 5$

$\Rightarrow \quad 18 \le f(8)-f(2) \le 30$

graphically: if $f(8)-f(2)$ is too big then there would be a point where $f'(c)$ is bigger than 5



if $f(8)-f(2)$ is too small then there would be a point where $f'(c)$ is less than 3



* Why? Since f differentiable on $(a,b) \Rightarrow f$ continuous on (a,b). See p 171.