Mat 135 Nov 1, 2004

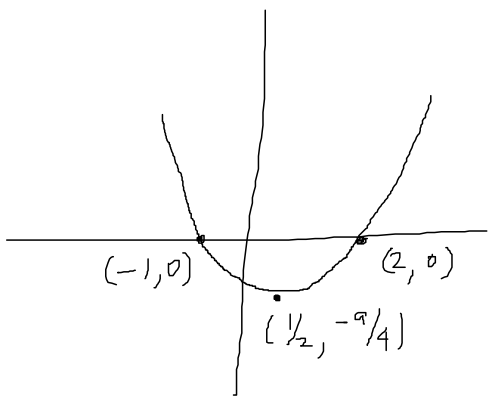
§ 3.7 Higher derivatives.

If f is differentiable then you can find f'. What if f' is differentiable? Well you can compute its derivative too!

$$(f')' = f'' = \text{second derivative of } f.$$

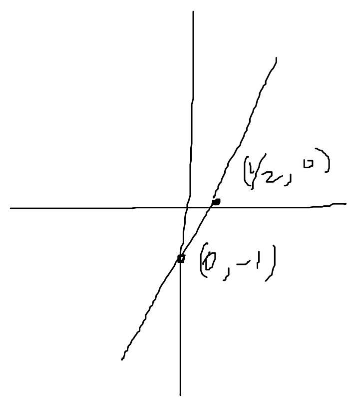
<u>For example:</u>

$$f(x) = (x+1)(x-2) \Rightarrow f'(x) = 2x-1 \Rightarrow f''(x) = 2$$



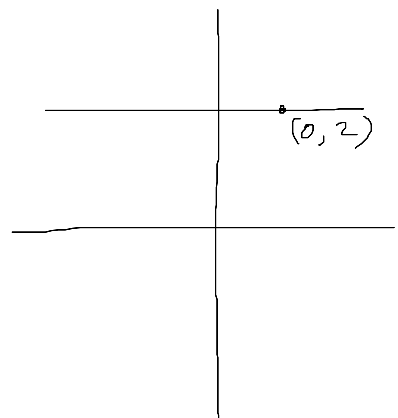
graph of f

expect $f' < 0$ for $x < 1/2$
$f' > 0$ for $x > 1/2$
$f' = 0$ at $x = 1/2$



graph of f'

expect $(f')' > 0$ for all x.



graph of f''

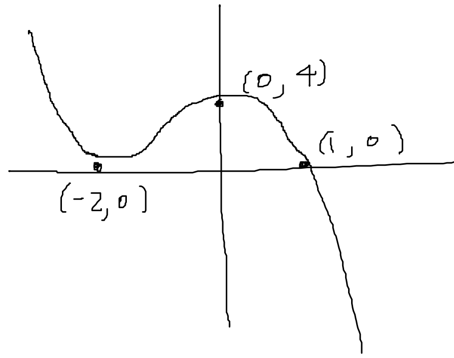
$(f')' = f''$ gives us information about how f' is changing. That is, if you graph f' then f'' tells you about the tangent line slopes in the graph of f'.

$f(x) = -(x+2)^2(x-1)$

from the graph,
expect

$f' < 0$ for $x < -2$ or $x > 0$

$f' > 0$ for $-2 < x < 0$



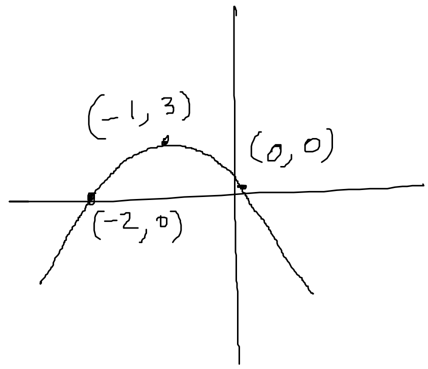
graph of f

$f'(x) = -3x(x+2)$

from the graph,
expect

$f'' > 0$ for $x < -1$

$f'' < 0$ for $x > -1$

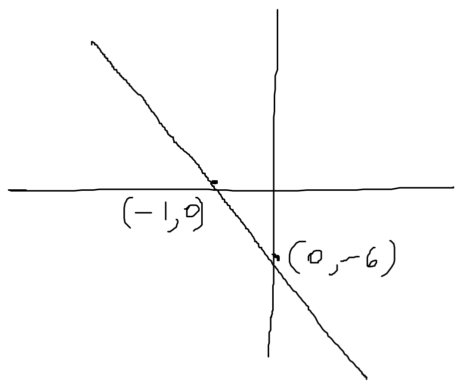


graph of f'

$f''(x) = -6(x+1)$

from the graph,
expect

f''' constant

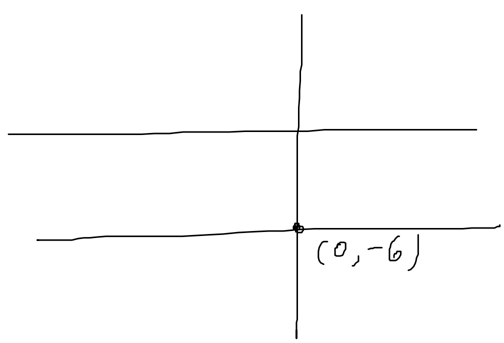


graph of f''

$f'''(x) = -6$

from the graph,
expect

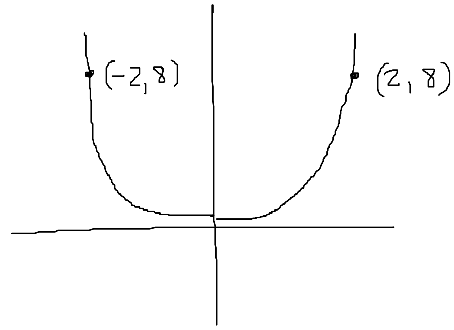
$f'''' = 0$



graph of f'''

$f''''(x) = 0$

$$f(x) = x^2|x| = \begin{cases} x^3 & \text{if } x \geq 0 \\ -x^3 & \text{if } x < 0 \end{cases}$$

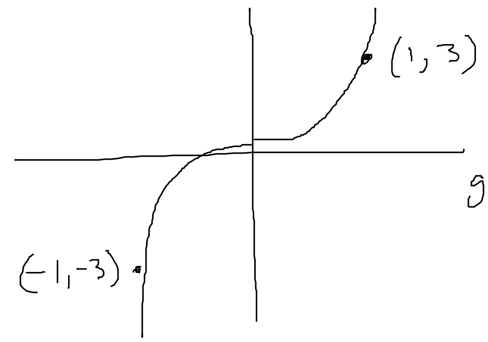


graph of f

from graph, expect
$f' < 0$ for $x < 0$
$f' > 0$ for $x > 0$

$$f'(x) = \begin{cases} 3x^2 & \text{if } x \geq 0 \\ -3x^2 & \text{if } x < 0 \end{cases}$$

$$= 3x|x|$$

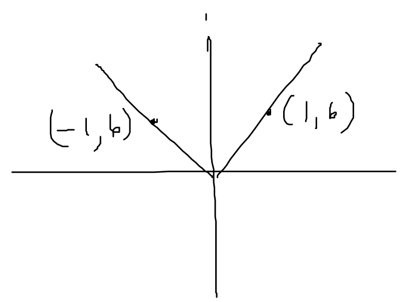


graph of f'

from graph, expect
$f'' > 0$ for $x < 0$ or $x > 0$

$$f''(x) = \begin{cases} 6x & \text{if } x \geq 0 \\ -6x & \text{if } x < 0 \end{cases}$$

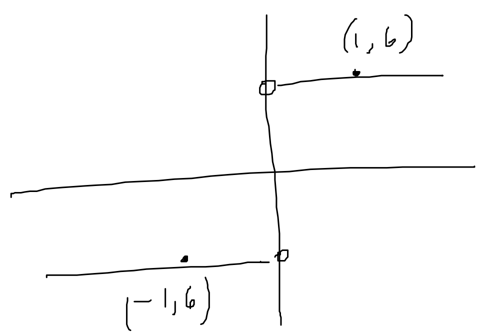
$$= 6|x|$$



graph of f''

from graph, expect
$f''' < 0$ for $x < 0$
$f''' > 0$ for $x > 0$
$f'''(0)$ does not exist

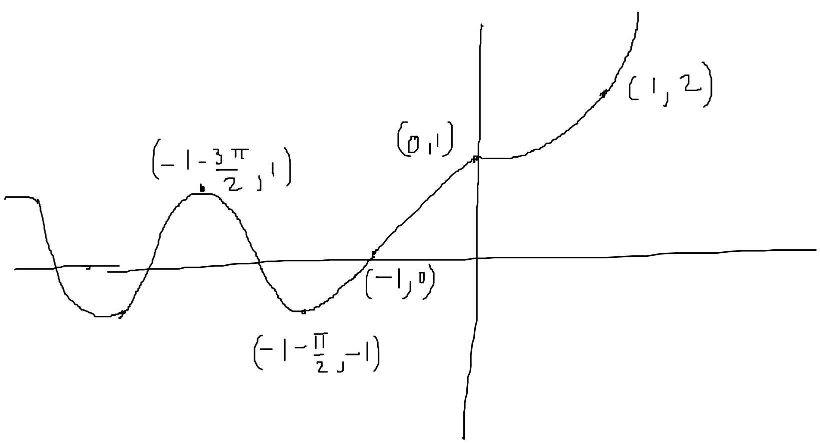
$$f''' = \begin{cases} 6 & \text{if } x > 0 \\ -6 & \text{if } x < 0 \end{cases}$$



graph of f'''

How to compute the derivative of a piecewise defined function...

$$f(x) = \begin{cases} x^2+1 & x \geq 0 \\ x+1 & -1 \leq x < 0 \\ \sin(x+1) & x < -1 \end{cases}$$



there are two points we have to worry about

$x=0$ (where the graphs of $x+1$ and x^2+1 meet)

$x=-1$ (where the graphs of $\sin(x+1)$ and $x+1$ meet)

everywhere else, we know f is differentiable.

So for $x>0$ $f'(x) = 2x$

for $-1<x<0$ $f'(x) = 1$

for $x<-1$ $f'(x) = \cos(x+1)$

check the two join points.

$x=0$ $\displaystyle\lim_{h\to 0+} \frac{f(0+h)-f(0)}{h} = 0$ $\displaystyle\lim_{h\to 0-} \frac{f(0+h)-f(0)}{h} = 1$

Since $\lim\limits_{h\to 0+} \frac{f(0+h)-f(0)}{h} \neq \lim\limits_{h\to 0-} \frac{f(0+h)-f(0)}{h}$

we know that $\lim\limits_{h\to 0} \frac{f(0+h)-f(0)}{h}$ does not exist,

so $f'(0)$ is not defined. (Note: you could do this by observing that $\lim\limits_{x\to 0+} f'(x) = 0 \neq \lim\limits_{x\to 0-} f'(x) = 1$)

Now check $x=-1$.

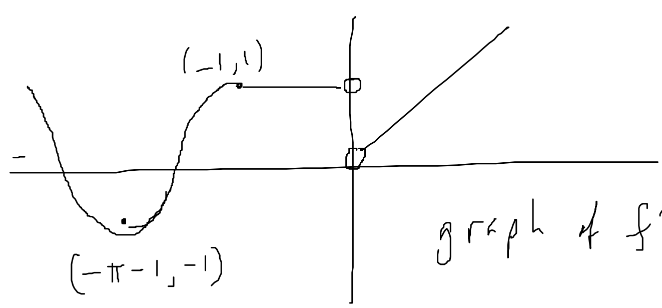
$$\lim_{h\to 0+} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0+} \frac{((-1+h)+1)-((-1)+1)}{h} = 1$$

$$\lim_{h\to 0-} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0-} \frac{\sin(h)-\sin(0)}{h} = 1$$

So $\lim\limits_{h\to 0} \frac{f(-1+h)-f(-1)}{h} = 1$ and $f'(-1)$ is defined.

We conclude.

$$f'(x) = \begin{cases} 2x & x>0 \\ 1 & -1\le x<0 \\ \cos(x+1) & x<-1 \end{cases}$$



Continuing in this manner,

$$f''(x) = 2 \quad \text{if } x > 0$$

$$f''(x) = 0 \quad \text{if } -1 < x < 0$$

$$f''(x) = -\sin(x+1) \quad \text{if } x < -1$$

there's only one extra point to check: $x = -1$

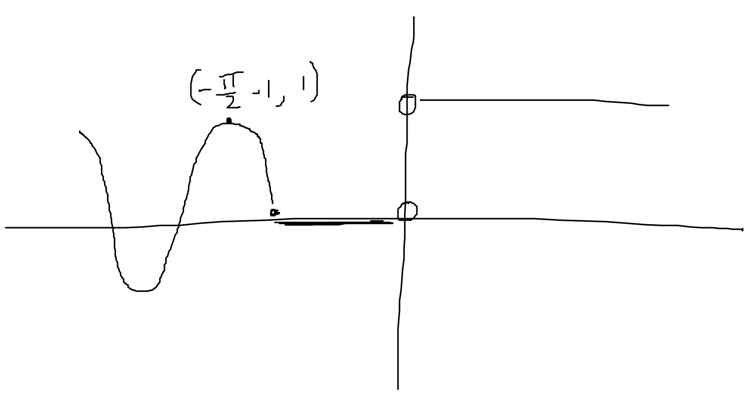
$$\lim_{h \to 0+} \frac{f'(-1+h) - f'(-1)}{h} = \lim_{h \to 0+} \frac{1-1}{h} = 0$$

$$\lim_{h \to 0-} \frac{f'(-1+h) - f'(-1)}{h} = \lim_{h \to 0-} \frac{\cos(h) - \cos(0)}{h} = 0$$

since $\lim_{h \to 0+} = \lim_{h \to 0-}$ we know $\lim_{h \to 0} \frac{f'(-1+h) - f'(-1)}{h}$

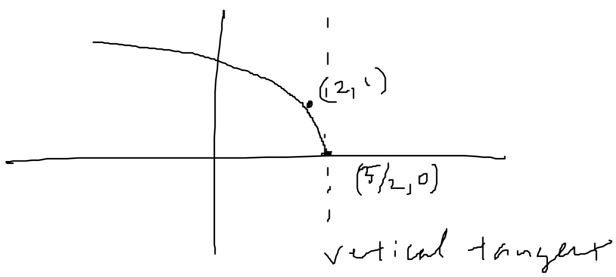
exists and $f''(-1) = 0$

we conclude:
$$f''(x) = \begin{cases} 2 & \text{if } x > 0 \\ 0 & \text{if } -1 \le x < 0 \\ -\sin(x+1) & \text{if } x < -1 \end{cases}$$



graph of f''

#26 if $g(x) = \sqrt{5-2x}$ find $g'''(2)$.



graph of g.

No problems with derivatives of g!

$$g(x) = (5-2x)^{1/2}$$

$$\Rightarrow g'(x) = \tfrac{1}{2}(5-2x)^{-1/2}(-2) = -(5-2x)^{-1/2}$$

$$\Rightarrow g''(x) = \tfrac{1}{2}(5-2x)^{-3/2}(-2) = -(5-2x)^{-3/2}$$

$$\Rightarrow g'''(x) = \tfrac{3}{2}(5-2x)^{-5/2}(-2) = -3(5-2x)^{-5/2}$$

so $g'''(-2) = -3(5-2\cdot 2)^{-5/2} = -3(+1)^{-5/2} = -3$

#40 find $D^{1000}\, xe^{-x}$

note: D is another way of writing $\frac{\partial}{\partial x}$.

We're ~~been~~ being asked for the 1000^{th} derivative of xe^{-x}. We'll do the first few derivatives and see if we can find a pattern.

$f(x) = x e^{-x}$

$\Rightarrow f'(x) = (1-x) e^{-x}$

$\Rightarrow f''(x) = (x-2) e^{-x}$

$\Rightarrow f'''(x) = (3-x) e^{-x}$

$\Rightarrow f''''(x) = (x-4) e^{-x}$

$\Rightarrow f'''''(x) = (5-x) e^{-x}$

n even $\Rightarrow D^n f = (x-n) e^{-x}$

n odd $\Rightarrow D^n f = (n-x) e^{-x}$

$$\Rightarrow \boxed{D^{1000} f = (x-1000) e^{-x}}$$

In general, $D^n f = (-1)^n (x-n) e^{-x}$

since $(-1)^{\text{even}} = 1$ and $(-1)^{\text{odd}} = -1$