Mat 135 March 9, 2005

§11.4 The comparison tests

The Comparison Test: Suppose that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms.

(i) If $a_n \le b_n$ for all $n$ and $\sum b_n$ converges then $\sum a_n$ converges

(ii) if $b_n \le a_n$ for all $n$ and $\sum b_n$ diverges then $\sum a_n$ diverges

The limit comparison test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

then $0 < c < \infty \Rightarrow$ both series converge or both series diverge.

ex: $\sum_{n=1}^{\infty} \frac{6}{7n^3+3n}$ conv or div?

by comparison test: $\frac{6}{7n^3+3n} < \frac{6}{7n^3}$ and

since $\sum \frac{6}{7n^3}$ converges we see $\sum \frac{6}{7n^3+3n}$ conv.

By limit comparison test, if

$$\sum a_n = \sum \frac{6}{7n^3+3n} \quad \text{and} \quad \sum b_n = \sum \frac{1}{n^3}$$

then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6}{7n^3+3n} \cdot \frac{n^3}{1}$

$$= \lim_{n\to\infty} \frac{6n^3}{7n^3+3n} = \frac{6}{7}$$

since $0 < \frac{6}{7} < \infty$ we see that both series will do the same thing. By the integral test, $\sum \frac{1}{n^3}$ converges, hence $\sum \frac{6}{7n^3+3n}$ converges.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n+\sqrt{n}}$ by comparison test we have 2 choices: $\displaystyle\frac{2}{3^n+\sqrt{n}} \le \frac{2}{3^n}$ and

$$\frac{2}{3^n+\sqrt{n}} \le \frac{1}{\sqrt{n}}$$

the first choice is useful since $\sum \frac{2}{3^n}$ converges. and so our series must converge. The second choice provides no information since the series $\sum \frac{1}{\sqrt{n}}$ diverges.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{2}{3^n - \sqrt{n}}$

method 1: limit comparison test with the series $\sum \frac{1}{3^n}$

$a_n = \dfrac{2}{3^n - \sqrt{n}} \qquad b_n = \dfrac{1}{3^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2}{3^n - \sqrt{n}} \cdot \frac{3^n}{1} = \lim_{n\to\infty} \frac{2\cdot 3^n}{3^n - \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{2}{1 - \sqrt{n}\, 3^{-n}} = 2$$

since $0 < 2 < \infty$ we see that both series do the same thing. And since $\sum \frac{1}{3^n}$ converges, we see that our series converges too.

method 2: direct comparison test.

$\dfrac{2}{3^n - \sqrt{n}} \leq \dfrac{4}{3^n}$ and since $\sum \frac{4}{3^n}$ converges, we see that our series converges.

But wait! How do I know $\dfrac{2}{3^n - \sqrt{n}} \leq \dfrac{4}{3^n}$ for all $n \geq 1$?

look at

$$f(x) = \frac{2}{3^x - \sqrt{x}} \quad \text{and} \quad g(x) = \frac{4}{3^x}$$

at $x=1$, $f(1) = \frac{2}{3^1 - \sqrt{1}} = 1$ $\quad g(1) = \frac{4}{3}$

so at $x=1$ $\quad f(x) < g(x)$.

is there some $x_0 > 1$ where $f(x_0) = g(x_0)$ ?
If there is no such $x_0$ then $f(x) < g(x)$ for all $x > 1$.

$$f(x_0) = g(x_0) \implies 2 \cdot 3^{x_0} = 4\left(3^{x_0} - \sqrt{x_0}\right)$$

$$\implies 4\sqrt{x_0} = 2 \cdot 3^{x_0}$$

this can't happen for $x_0 > 1$
because

$$4\sqrt{1} < 2 \cdot 3^1$$

and

$4\sqrt{x}$ increases more slowly than $2 \cdot 3^x$

so $4\sqrt{x}$ starts out behind $2 \cdot 3^x$ and can never catch up with it. So there's no $x_0$ so that $f(x_0) = g(x_0)$

$$\implies \frac{2}{3^n - \sqrt{n}} \le \frac{4}{3^n} \quad \text{for all } n \geq 1$$

<u>real lesson:</u> If you don't mind taking limits, the limit comparison test can be far more useful!!

<u>ex:</u> $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$ do the LCT with the series $\displaystyle\sum \frac{n}{n^{7/3}} = \sum \frac{1}{n^{4/3}}$

Since $\displaystyle\lim_{n\to\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}} \cdot \frac{n^{4/3}}{1} = 1$ and $0<1<\infty$

we see that our series converges since $\displaystyle\sum \frac{1}{n^{4/3}}$ converges.

<u>ex:</u>
$$\sum_{n=1}^{\infty} \frac{3n^2-5n}{n^3+n+1} = \frac{3\cdot 1^2 - 5\cdot 1}{1^3+1+1} + \sum_{n=2}^{\infty} \frac{3n^2-5n}{n^3+n+1}$$
$$= \frac{-2}{3} + \sum_{n=2}^{\infty} \underbrace{\frac{3n^2-5n}{n^3+n+1}}_{\text{are } >0 \text{ for } n\geq 2}$$

do limit comparison test with $\displaystyle\sum \frac{1}{n}$ and find that $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 3$

Since $0<3<\infty$ the LCT $\Rightarrow$ our series <u>diverges</u>

ex:

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} = \sum_{n=1}^{\infty} \frac{1}{n \cdot n^{1/n}}$$

do limit comparison test with $\sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n \cdot n^{1/n}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$$

Since $0 < 1 < \infty$ the LCT says that both series do the same thing. Since $\sum \frac{1}{n}$ diverges, our series diverges.