Mat 135 March 7, 2005

If you go back at look at the notes on log where I proved that

$$\lim_{x \to \infty} \int_1^x \frac{1}{t}\,dt = \infty$$

You'll find that the proof of divergence is the exact same as the proof that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

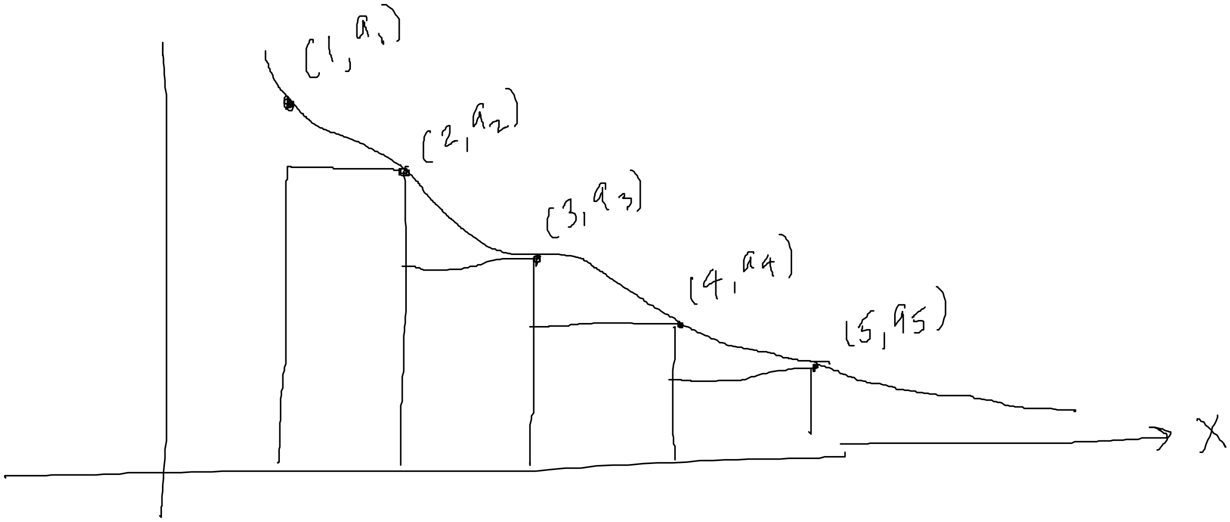
In fact, improper integrals are sometimes very closely related to series.

Thm: (the Integral test) Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n := f(n)$.

a) if $\int_1^{\infty} f(x)\,dx$ is convergent then $\sum_{n=1}^{\infty} a_n$ is too.

b) if $\int_1^{\infty} f(x)\,dx$ is divergent then $\sum_{n=1}^{\infty} a_n$ is too

Why would this be true?



$$S_5 = a_1 + a_2 + a_3 + a_4 + a_5 < a_1 + \int_1^5 f(x)\,dx$$

in general, $S_n < a_1 + \int_1^n f(x)\,dx$.

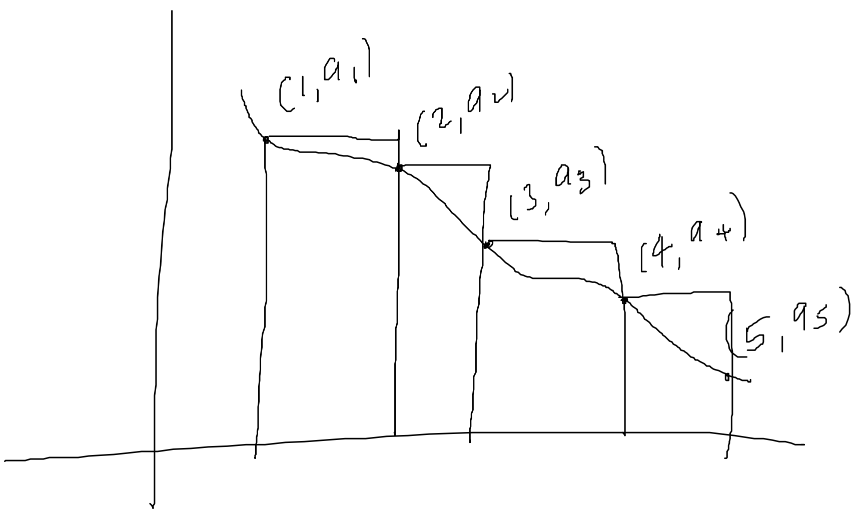
If $\int_1^\infty f(x)$ is convergent then the right-hand side $a_1 + \int_1^n f(x)\,dx \le a_1 + \int_1^\infty f(x)\,dx < \infty$. And

so $\{S_n\}_1^\infty$ is an increasing sequence that is bounded above. By the Monotone Sequence Theorem of §11.1, $\lim_{n\to\infty} S_n$ exists and is finite $\Rightarrow \sum_{n=1}^\infty a_n$ is convergent

This shows that

$$\int_1^\infty f(x)\,dx \text{ convergent} \Rightarrow \sum_{n=1}^\infty a_n \text{ convergent.}$$

Now consider the other case:



Here, you see that

$$S_4 = a_1 + a_2 + a_3 + a_4 > \int_1^5 f(x)\,dx$$

In general, $S_n > \int_1^{n+1} f(x)\,dx$. If $\int_1^\infty f(x)$ is divergent, then $\int_1^{n+1} f(x) \to \infty$ as $n \to \infty$. This forces $S_n \to \infty$ as $n \to \infty$. Hence $\sum_{n=1}^\infty a_n$ is divergent.

From the section on improper integrals, we know $\int_1^\infty \frac{1}{x^p}\,dx$ converges if $p>1$ and diverges if $p \le 1$. It follows immediately that

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if } p > 1 \quad \text{and}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ diverges if } p \le 1$$

<u>ex:</u> $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Why? take $p=1$

$\sum_{n=1}^{\infty} \frac{1}{n^{1.001}}$ converges. Why? $p = 1.001 > 1$.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges since $p = 1/2 \le 1$.

$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges since $\int_2^\infty \frac{1}{x\ln(x)}\,dx$ diverges

since $$\int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{n\to\infty} \int_2^n \frac{1}{x\ln(x)}\,dx = \lim_{n\to\infty} \ln(\ln(n)) - \ln(\ln(2))$$

$\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges. Why? Since $\int_2^{\infty} \frac{1}{\ln(x)}\,dx$

diverges. Why? Since $\ln(x) < x \Rightarrow \frac{1}{x} < \frac{1}{\ln(x)}$

$\Rightarrow \int_2^{\infty} \frac{1}{x}$ divergent $\Rightarrow \int_2^{\infty} \frac{1}{\ln(x)}$ divergent

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{\ln(n)}}$ diverges? Why? Since $\int_2^{\infty} \frac{1}{\sqrt{\ln(x)}}\,dx$

diverges. Why? Since $\ln(x) < x$

$\Rightarrow \sqrt{\ln(x)} < \sqrt{x} \Rightarrow \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{\ln(x)}}$

and $\int_2^{\infty} \frac{1}{\sqrt{x}}\,dx$ divergent $\Rightarrow \int_2^{\infty} \frac{1}{\sqrt{\ln(x)}}\,dx$ divergent ex: $$\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)} = \sum_{n=1}^{\infty}\left[\frac{2}{n} + \frac{1}{n+1}\right]$$

← don't hope for a telescoping sequence... it's all sums, no subtractions

diverges since

$$\int_1^{\infty} \frac{2}{x} + \frac{1}{x+1}\,dx \text{ diverges.}$$

ex:

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} \text{ converges. since } \int_1^{\infty} \frac{\ln(x)}{x^2}\,dx$$

$$= \lim_{N\to\infty} \int_1^{N} \frac{\ln(x)}{x^2}\,dx = \lim_{N\to\infty} -\frac{\ln(N)}{N} - \frac{1}{N} + 1 = 1$$

note: In order to apply the integral test, I need to know that $\frac{\ln(x)}{x^2}$ is a decreasing function

$$f'(x) = \frac{1-2\ln(x)}{x^2} < 0 \quad \text{on } [2,\infty)$$

So we can apply the integral test to

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2} \quad \text{via} \quad \int_2^{\infty} \frac{\ln(x)}{x^2}\,dx.$$