Mat 135 March 4, 2005

§ 11.2 Series

Can one ever add up infinitely many numbers? Clearly this is impossible if one is using a calculator and don't have infinitely much time. On the other hand, we can sometimes figure out what number we'd get as the sum, if we could add those numbers.

<u>defn</u>: Given a series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \dots$$

Let S_n be the n^{th} <u>partial sum</u>

$$S_n := a_1 + a_2 + a_3 + \dots + a_n.$$

If $\lim_{n\to\infty} S_n$ exists and is finite then we say the series is <u>convergent</u> and $\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} S_n$.

If the limit doesn't exist or is $\pm\infty$ then the series is called <u>divergent</u>

ex: the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} \qquad \text{where } a \neq 0$$

$S_1 = a$

$S_2 = a + ar$

$S_3 = a + ar + ar^2$

$\vdots$

$$S_n = a + ar + ar^2 + \dots + ar^{n-1} = a(1 + r + r^2 + \dots + r^{n-1})$$

$$= \frac{a(1-r^n)}{1-r} \qquad \boxed{\text{if } r \neq 1}$$

So to see if $\sum_{n=1}^{\infty} ar^{n-1}$ is convergent or divergent

we need to understand

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} a\frac{1-r^n}{1-r} = \begin{cases} \frac{a}{1-r} & \text{if } |r|<1 \\ \operatorname{sgn}(a)\infty & \text{if } r>1 \\ \text{DNE} & \text{if } r<-1 \end{cases}$$

recall

$$\operatorname{sgn}(a) = \begin{cases} +1 & a>0 \\ -1 & a<0 \end{cases}$$

if $r=1$ then $S_n = \underbrace{a + a + a + \dots + a}_{n \text{ times}} = na$

and $\lim_{n\to\infty} S_n = \infty$

Similarly, if $r = -1$ then

$S_1 = a$

$S_2 = a - a = 0$

$S_3 = a - a + a = a$

$S_4 = a - a + a - a = 0$

$\vdots$

$$S_n = \begin{cases} a & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$\lim_{n \to \infty} S_n$ DNE.

to conclude; $\lim_{n\to\infty} S_n$ exists and is finite only if $|r| < 1$. In this case, $\lim_{n\to\infty} S_n = \frac{a}{1-r}$ and

so
$$\sum_{n=1}^{\infty} a r^{n-1} = \begin{cases} \frac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & \text{if } |r| \geq 1 \end{cases}$$

<u>ex:</u> $\displaystyle\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1} = \frac{5}{1 - 2/3} = \frac{5}{1/3} = 15$

<u>ex:</u> $\displaystyle\sum_{n=1}^{\infty} \frac{(-6)^{n-1}}{5^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{-6}{5}\right)^{n-1}$ divergent

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty} e\,\frac{e^{n-1}}{3^{n-1}} = \sum_{n=1}^{\infty} e\left(\frac{e}{3}\right)^{n-1} = \frac{e}{1-e/3}$

$\displaystyle = \frac{3e}{3-e}$

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{\pi^n}{3^{n+1}} = \sum_{n=1}^{\infty} \frac{\pi\,\pi^{n-1}}{3^2\,3^{n-1}} = \sum_{n=1}^{\infty} \frac{\pi}{9}\left(\frac{\pi}{3}\right)^{n-1}$

divergent

Sometimes series will have nice telescoping properties for example

$$\sum_{n=1}^{\infty} \frac{2}{n^2+2n} = \sum_{n=1}^{\infty}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

$S_1 = \frac{1}{1} - \frac{1}{3}$

$S_2 = S_1 + \frac{1}{2} - \frac{1}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$

$S_3 = S_2 + \frac{1}{3} - \frac{1}{5} = \frac{1}{1} + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$

$S_4 = S_3 + \frac{1}{4} - \frac{1}{6} = \frac{1}{1} + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$

$S_5 = S_4 + \frac{1}{5} - \frac{1}{7} = \frac{1}{1} + \frac{1}{2} - \frac{1}{6} - \frac{1}{7}$

$S_6 = S_5 + \frac{1}{6} - \frac{1}{8}$

$= \frac{1}{1} + \frac{1}{2} - \frac{1}{7} - \frac{1}{8}$

and so on

we see that

$$S_n = \frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \quad \text{for } n \geq 2$$

Hence $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right) = 3/2$

Since the partial sums have a finite limit, the series is <u>convergent</u> and

$$\sum_{n=1}^{\infty} \frac{2}{n^2+2n} = 3/2$$

these telescoping miracles aren't common. When they happen, it's usually because a_n = rational function and one has to write a_n as a partial fraction to see the telescoping in action.

<u>Theorem</u>: If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$

<u>corollary</u>: if $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty}$ is divergent

<u>ex</u>: $\sum_{k=1}^{\infty} \frac{k^2}{k^2-1}$ divergent since $\lim_{k\to\infty} a_k = 1 \neq 0$

ex: $\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$ is divergent since $\lim_{n\to\infty} a_n = \ln(1/2) \neq 0$.

Note: It does not follow that

$$\lim_{n\to\infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n \text{ is convergent.}$$

If only life were so easy!!!

For example, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. This is the famous Harmonic series. It diverges for the same reason that $\int_1^{\infty} \frac{1}{x}\,dx$ diverges.

$$S_1 = \frac{1}{1}$$

$$S_2 = \frac{1}{1} + \frac{1}{2}$$

$$S_4 = \frac{1}{1} + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > \frac{1}{1} + \frac{1}{2} + \frac{1}{2} = \frac{1}{1} + \frac{2}{2}$$

$$S_8 = S_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > S_4 + (1/8 + 1/8 + 1/8 + 1/8)$$

$$> S_4 + 1/2 > 1/1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{1}{1} + \frac{3}{2}$$

In general

$$S_{2^n} > 1 + \frac{n}{2}$$

This shows that $\lim_{n\to\infty} S_{2^n} = \infty$ hence $\lim_{n\to\infty} S_n = \infty$

and so $\sum_{n=1}^{\infty} \frac{1}{n}$ is <u>divergent</u>.

Finally, we have the usual rules for sums & differences & stuff

<u>Theorem</u>: If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent then so are $\sum_{n=1}^{\infty} ca_n$, $\sum_{n=1}^{\infty} a_n+b_n$, and $\sum_{n=1}^{\infty} a_n-b_n$

and

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

$$\sum_{n=1}^{\infty} a_n+b_n = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

$$\sum_{n=1}^{\infty} a_n-b_n = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$$

<u>ex</u>: $$\sum_{n=1}^{\infty} \frac{3^n+2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3}{6}\right)^n + \left(\frac{2}{6}\right)^n = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} + \frac{1}{3}\left(\frac{1}{3}\right)^{n-1}$$

$$\underset{\text{theorem!}}{=} \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{1}{3}\right)^{n-1} = \frac{1/2}{1-1/2} + \frac{1/3}{1-1/3}$$

8

ex: Express the # as a ratio of integers.

$6.2\overline{54}$

$= 6.2 + .05454545454\overline{54}$

$= \frac{62}{10} + \frac{54}{1{,}000} + \frac{54}{100{,}000} + \frac{54}{10000000} + \cdots$

$= \frac{62}{10} + \frac{54}{10^3} + \frac{54}{10^5} + \frac{54}{10^7} + \cdots$

$= \frac{62}{10} + \sum_{n=1}^{\infty} \frac{54}{10^{2n+1}}$

$= \frac{62}{10} + \sum_{n=1}^{\infty} \frac{54}{10\cdot(100)^n} = \frac{62}{10} + \sum_{n=1}^{\infty} \frac{54}{10(100)\,100^{n-1}}$

$= \frac{62}{10} + \sum_{n=1}^{\infty} \frac{54}{1000}\left(\frac{1}{100}\right)^{n-1}$

$= \frac{62}{10} + \frac{54/1000}{1 - 1/100}$

$= \boxed{\frac{344}{55}}$