Mat 135 March 2, 2005

given a sequence we would like to know if $\lim_{n\to\infty} a_n$ exists.

Theorem: if you can find a function $f:[1,\infty)\to\mathbb{R}$ so that $f(n)=a_n$ for all n then if $\lim_{x\to\infty} f(x)$ exists this implies $\lim_{n\to\infty} a_n$ exists and $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$.

ex: $a_n = \ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right)$

then since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0$

we see $\lim_{n\to\infty} a_n = 0$

ex: $a_n = n\sin\left(\frac{1}{n}\right)$ Using $f(x) = x\sin\left(\frac{1}{x}\right)$

$$\lim_{x\to\infty} x\sin(1/x) = \lim_{x\to\infty} \frac{\sin(1/x)}{1/x} = \lim_{y\to 0+} \frac{\sin(y)}{y}$$

$$= \lim_{y\to 0+} \frac{\cos(y)}{1} = 1$$ used l'Hospital.

Hence $\lim_{n\to\infty} a_n = 1$

ex: $a_n = \sqrt{n} - \sqrt{n^2 - 1}$

take $f(x) = \sqrt{x} - \sqrt{x^2-1}$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\sqrt{x} - \sqrt{x^2-1}\right) = \lim_{x\to\infty}\left(\sqrt{x} - x\sqrt{1-1/x^2}\right)$$

$$= \lim_{x\to\infty} x\left(\frac{1}{\sqrt{x}} - \sqrt{1-1/x^2}\right)$$

$$= -\infty$$

Hence $\lim_{n\to\infty} a_n = -\infty$

ex: $a_n = \frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} + \frac{4}{n^2} + \cdots + \frac{n}{n^2}$

$$= \frac{\sum_{i=1}^{n} i}{n^2} = \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$$

since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

take $f(x) = \frac{x(x+1)}{2x^2}$ and find $\lim_{n\to\infty} a_n = 1/2$

Just as we had limit laws for functions, we have limit laws for sequences.

And we also have a squeeze theorem:

if $a_n \le b_n \le c_n$ for all n and

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ then $\lim_{n\to\infty} b_n$ exists

and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n$.

ex: $a_n = \dfrac{\sin(2n)}{1+\sqrt{n}}$

use the squeeze theorem since

$$\frac{-1}{1+\sqrt{n}} \le \frac{\sin(2n)}{1+\sqrt{n}} \le \frac{1}{1+\sqrt{n}}$$

$$\lim_{n\to\infty} \frac{-1}{1+\sqrt{n}} = 0 \quad \& \quad \lim_{n\to\infty} \frac{1}{1+\sqrt{n}} = 0 \Rightarrow \lim_{n\to\infty} \frac{\sin(2n)}{1+\sqrt{n}} = 0$$

by the squeeze theorem

ex: $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \dots\right\}$ $\lim_{n\to\infty} a_n = ?$

a_n = one thing if n odd, another if n even

$n = 1 \Rightarrow 2n = 2$
$n = 2 \Rightarrow 2n = 4$
$n = 3 \Rightarrow 2n = 6$

$\Rightarrow a_{2n} = \dfrac{1}{n+2}$

odd cases? $n=1 \rightarrow 2n-1=1$

$n=2 \rightarrow 2n-1=3$

$n=3 \rightarrow 2n-1=5$

etc

$$\rightarrow a_{2n-1} = \frac{1}{n}$$

We've figured out that the odd elements can be written as

$a_{2n-1} = 1/n \Rightarrow$ if m is odd then $a_m = \frac{2}{m+1}$

(why? $m = 2n-1$
$\Rightarrow m+1 = 2n$
$\Rightarrow \frac{m+1}{2} = n$
$\Rightarrow \frac{2}{m+1} = 1/n$)

even elements can be written as

$a_{2n} = \frac{1}{n+2} \Rightarrow$ if m is even then $a_m = \frac{2}{m+4}$

to sum up, $a_m = \begin{cases} \frac{2}{m+1} & \text{if } m \text{ odd} \\ \frac{2}{m+4} & \text{if } m \text{ even} \end{cases}$

Now that we have a formula for a_m, we can apply the squeeze theorem

Specifically

$$0 \le a_n \le \frac{2}{n} \quad \text{for all } n.$$

And so the squeeze theorem implies

$$\lim_{n \to \infty} a_n = 0$$ (which is what

you expected just by inspection! ☺)

ex: $a_n = \dfrac{n!}{2^n}$

$a_1 = \dfrac{1}{2}$ $\quad a_2 = \dfrac{2 \cdot 1}{2 \cdot 2}$ $\quad a_3 = \dfrac{3 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2}$ $\quad a_4 = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdot 2}$

$a_5 = \dfrac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2}$

note that $a_2 = 1/2$ $\quad a_3 = \frac{3}{2} \cdot \frac{1}{2}$ $\quad a_4 > \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}$

$a_5 > \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}$ and so on. So

$$a_n > \left(\frac{3}{2}\right)^{n-2} \frac{1}{2} \quad \text{for all } n \geq 4$$

Since the sequence $b_n = \left(\frac{3}{2}\right)^{n-2} \frac{1}{2}$ diverges to infinity, this will force a_n to diverge to ∞ too.

Another useful theorem is

Theorem: if $\lim_{n\to\infty} |a_n| = 0$ then $\lim_{n\to\infty} a_n = 0$

This helps with sequences that change sign in strange ways:

ex: $\lim_{n\to\infty} \frac{\sin(2n)}{1+\sqrt{n}} = 0$. Why?

$\lim_{n\to\infty} \left|\frac{\sin(2n)}{1+\sqrt{n}}\right| \le \lim_{n\to\infty} \frac{1}{1+\sqrt{n}}$ since $|\sin(2n)| \le 1$

and so by the squeeze theorem

applied to $\left|\frac{\sin(2n)}{1+\sqrt{n}}\right|$ $\left(0 \le \left|\frac{\sin(2n)}{1+\sqrt{n}}\right| \le \frac{1}{1+\sqrt{n}} \right)$

we get $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$.

Here's one theorem that we'll use <u>a lot</u>

Theorem: every bounded monotonic sequence is convergent.

What does monotonic mean?

<u>defn</u>: A sequence $\{a_n\}$ is <u>increasing</u> if $a_n < a_{n+1}$ for all $n \geq 1$. It is <u>decreasing</u> if $a_n > a_{n+1}$ for all $n \geq 1$. A sequence is <u>monotonic</u> if it is either increasing or decreasing

<u>ex</u>: $\{a_n\} = \{0, 1, 0, 1, 0, 1, 0, 1 \ldots\}$ is not monotonic

<u>ex</u>: $\{a_n\} = \cos(n)$ is not monotonic

<u>ex</u>! $a_n = 3 - \frac{1}{n+5}$ is increasing hence is monotonic. Why is it increasing?

We need to show $a_n < a_{n+1}$ for all $n \geq 1$.

$$a_n = 3 - \frac{1}{n+5} \overset{?}{<} 3 - \frac{1}{(n+1)+5} = 3 - \frac{1}{n+6} = a_{n+1}$$

this is true if $\frac{-1}{n+5} < -\frac{1}{n+6}$.

Which is true if $-(n+6) < -(n+5)$. This is always true because

$-n-6 < -n-5$ since $-6 < -5$. ✓

ex: $a_n = ne^{-n}$ is decreasing hence monotonic.

Why? Method 1:

try to show that

$$a_n = ne^{-n} > (n+1)e^{-(n+1)} = a_{n+1}$$

since

$$(n+1)e^{-(n+1)} = (n+1)e^{-n}e^{-1}$$
$$= ne^{-n}e^{-1} + e^{-n}e^{-1}$$
$$= e^{-1}\left[ne^{-n} + e^{-n}\right]$$

we see that

$$a_{n+1} = e^{-1}\left[ne^{-n} + e^{-n}\right] < e^{-1}\left[ne^{-n} + ne^{-n}\right]$$
$$= e^{-1}\left[2a_n\right]$$
$$= \frac{2}{e}a_n$$

So $a_{n+1} < \frac{2}{e}a_n < a_n$. Ouch! that was a lot of work!

Alternately, since $a_n = f(n)$ where $f(x) = xe^{-x}$ we see that if we can show that f is a decreasing function of x then $f(n) > f(n+1) \Rightarrow a_n > a_{n+1}$ as desired.

$$f(x) = xe^{-x} \Rightarrow f'(x) = (1-x)\,e^{-x} < 0$$

for $x > 1$

Hence $f(1) > f(2) > f(3) \dots$

ex: $a_n = n + \frac{1}{n}$

same idea as before: $f(x) = x + \frac{1}{x}$ then

$$f'(x) = 1 - \frac{1}{x^2} > 0 \quad \text{for } x > 1$$

Hence $f(1) < f(2) < f(3) < \dots$

this shows $\{a_n\}$ is an increasing sequence.

Now for the theorem

<u>ex</u>: $\lim_{n\to\infty} \frac{1}{2n+3} = 0.$

Why? $0 \le a_n \le 1$ for all n,
So the sequence is bounded.

and if $f(x) = \frac{1}{2x+3}$ then $f'(x) = \frac{-2}{(2x+3)^2} < 0$

So the sequence $\{a_n\}$ is decreasing, hence it's monotonic. Thus by the theorem, $\lim_{n\to\infty} \frac{1}{2n+3}$ exists. (We know by other methods that the limit is zero)