Mat 135 March 16, 2005

Now that we see that proving that a series converges absolutely is a good goal to have, here are two tests that are useful.

ratio test:

1) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$ then $\sum a_n$ is absolutely convergent

2) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

then $\sum a_n$ is divergent.

3) if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$ then the ratio test gives no information

ex: $\sum_{n=1}^{\infty} \frac{1}{n!}$ then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)!}\cdot\frac{n!}{1}\right|$

$$= \lim_{n\to\infty} \frac{1}{n+1} = 0 < 1.$$

So $\sum \frac{1}{n!}$ is convergent.

Note! You can apply the ratio test to positive series if you want.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{\cancel{n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(n+1)}{(n+1)^{n+1}}\cdot n^n\right|$$

$$= \lim_{n\to\infty}\left|\frac{n^n}{(n+1)^n}\right| = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

by l'Hospital

Since $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$ we conclude that the series converges.

note: We did both these series earlier using the comparison test. That took more work!

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n}{3n^3-n^2+1}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^2+(n+1)}{3(n+1)^3-(n+1)^2+1}\cdot\frac{3n^3-n^2+1}{n^2+n}\right|$$

$$= 1$$

So the ratio test gives no information!

by the limit comparison test this series diverges.

(compare to $\Sigma \frac{1}{n}$)

The root test:

1) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely

2) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\sqrt[n]{|a_n|} \to \infty$

then $\sum a_n$ is divergent.

3) if $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ then the root test is inconclusive

ex: $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

$$= \lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1$$

so the series converges

ex: $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan(n))^n}$

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(-1)^n}{(\arctan(n))^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\arctan(n))^n}}$$

$$= \lim_{n\to\infty} \frac{1}{\arctan(n)} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1$$

so the series converges.

ex. $\sum_{n=1}^{\infty} \frac{n}{2^n}$

by ratio test: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n}$

$= \lim_{n\to\infty} \frac{1}{2} \frac{n+1}{n} = 1/2 < 1$

converges

by root test: $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{n}{2^n}}$

$= \lim_{n\to\infty} \frac{\sqrt[n]{n}}{2} = \frac{1}{2}$

l'Hospital

converges

§11.7 Strategies for testing series.

rule of thumb:

if a_n is algebraic, try the limit comparison test

if a_n has a factorial, try the ratio test

if a_n has something to the n^{th} power
and has no factorials, try the root test

if a_n has $\ln(n)$ in it, try comparison test
or integral test.

ex: $\displaystyle\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$

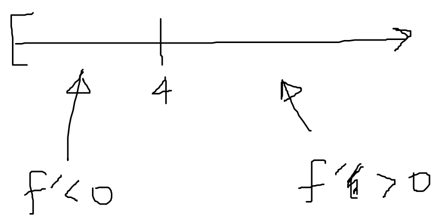
I know that $\ln(k)$ goes to infinity slower than to any power. For example,

$$\ln(k) \le \sqrt{k} \quad \text{for all } k \geq 1.$$

Why? let $f(x) = \sqrt{x} - \ln(x)$.

then $f(1) = 1$ and $f'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{x}$

$f'(1) < 0$ and $f'(x_0) = 0$ at $x_0 = 4$



$f(4) > 0$ and so $f(x) > 0$ for all $x \geq 1$

This shows that $f(k) > 0$ for all $k \geq 1$

$\Rightarrow \sqrt{k} > \ln(k)$ for all $k \geq 1$

$$\frac{k\ln(k)}{(k+1)^3} \le \frac{k\sqrt{k}}{(k+1)^3} \quad \text{for all } k \geq 1$$

so if $\sum_{k=1}^{\infty} \frac{k\sqrt{k}}{(k+1)^3}$ converges then $\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3}$ converges.

Now, applying the limit comparison test with $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$ we see that

$$\sum_{k=1}^{\infty} \frac{1}{k^{3/2}} \text{ converges (by p-test)}$$

$$\Downarrow$$

$$\sum_{k=1}^{\infty} \frac{k\sqrt{k}}{(k+1)^3} \text{ converges (by limit comparison test)}$$

$$\Downarrow$$

$$\sum_{k=1}^{\infty} \frac{k\ln(k)}{(k+1)^3} \text{ converges (by comparison test)}$$

ex: $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ try root test!

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{(2n)^n}{n^{2n}}\right|}$$

$$= \lim_{n\to\infty} \frac{2n}{n^n} = \lim_{n\to\infty} \frac{2}{n^{n-1}} = 0 < 1$$

so the root test implies convergence.

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

try limit convergence test with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{e^{1/n}}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} e^{1/n} = 1 < \infty$$

the series converges

ex: $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$

since $4^n > 3^n$

$4^n + 4^n > 3^n + 4^n = 2\,4^n$

so $\dfrac{1}{2 \cdot 4^n} < \dfrac{1}{3^n + 4^n}$

and $\dfrac{5^n}{2 \cdot 4^n} < \dfrac{5^n}{3^n + 4^n}$ for all n

By the comparison test with

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{2} \frac{5^n}{4^n}$ we see that $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{3^n + 4^n}$ diverges

diverges since $\left|\frac{5}{4}\right| > 1$. (geometric series)