§11.6 Absolute Convergence, Ratio Test, Root Test

Until now, we've considered series where all terms are positive or where the terms alternate in sign. What if some terms are positive and some are negative, but it's not alternating? This is where absolute & conditional convergence become important.

defn: A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent

defn: A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

ex: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$ are absolutely convergent

ex: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n \ln(n)}$ are conditionally convergent.

Is it possible that $\sum |a_n|$ might converge but $\sum a_n$ diverges? No. Absolute convergence is stronger than convergence:

<u>theorem:</u> If $\sum_{n=1}^{\infty} a_n$ converges absolutely then $\sum_{n=1}^{\infty} a_n$ converges.

<u>Transl:</u> $\sum_{n=1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

Why is this theorem useful?

<u>ex:</u> $\sum_{n=1}^{\infty} \cos(n) \frac{7n^2+1}{8n^5-3n}$ converges? diverges?

$$a_n = \underbrace{\cos(n)}_{\text{sometimes positive, sometimes negative, never zero.}} \underbrace{\frac{7n^2+1}{8n^5-3n}}_{\text{positive}}$$

The series is not alternating, so we cannot apply the alternating series test.

By the theorem, if $\sum_{n=1}^{\infty} |a_n|$ converges then we know $\sum_{n=1}^{\infty} a_n$ converges. Maybe we can show that $\sum_{n=1}^{\infty} |a_n|$ converges!

$$|a_n| = |\cos(n)| \left| \frac{7n^2+1}{8n^5-3n} \right| = |\cos(n)| \frac{7n^2+1}{8n^5-3n}$$

if we're going to apply the comparison test or the limit comparison test, we need to know that $|a_n| > 0$ for all n. Is this true?

$|a_n| > 0 \iff |\cos(n)| > 0$ for all n. (Since $7n^2+1 \neq 0$ for any n.)

$\iff \cos(n) \neq 0$ for all n

Could it happen that $\cos(n_0) = 0$ for some n_0?

If this were true then we'd have that $n_0 = \pi k$ for some integer k. $\Rightarrow \frac{n_0}{k} = \pi$. But this is impossible because $\frac{n_0}{k}$ is a rational number and π is an irrational number. So $|a_n| > 0$ and we can use comparison tests!

$$|a_n| = |\cos(n)| \frac{7n^2+1}{8n^5-3n} \leq \frac{7n^2+1}{8n^5-3n} =: b_n \quad \text{for all } n.$$

So by the comparison test,

if $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{7n^2+1}{8n^5-3n}$ converges then $\sum_{n=1}^{\infty} |a_n|$ converges

Does $\sum_{n=1}^{\infty} b_n$ converge? We apply the limit comparison test with the series $\sum_{n=1}^{\infty} c_n := \sum_{n=1}^{\infty} \frac{1}{n^3}$.

$$\lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{7n^2+1}{8n^5-3n} \cdot \frac{n^3}{1} = \frac{7}{8}$$

Since $0 < \lim_{n\to\infty} \frac{b_n}{c_n} < \infty$ $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ do the same thing.

By the p-test, $\sum_{n=1}^{\infty} c_n$ converges. Hence by the limit comparison test $\sum_{n=1}^{\infty} b_n$ converges.
Hence by the comparison test $\sum_{n=1}^{\infty} |a_n|$ converges.
Hence by the theorem $\sum_{n=1}^{\infty} a_n$ converges.

using similar arguments,

$$\sum_{n=1}^{\infty} \sin(\sqrt{7}\,n) \frac{1}{n \ln(n)^2} \text{ converges}$$

The arguments are useless for

$$\sum_{n=1}^{\infty} \cos(3n) \frac{1}{n \ln(n)}$$

however.