some more comments on comparison test vs limit comparison test

consider $\sum_{n=1}^{\infty} \frac{6}{7n^3+3n}$ if you want to apply the limit comparison test, you need to get the $n \to \infty$ behaviour of a_n just right. ie you need to recognize that $a_n \sim \frac{1}{n^3}$ as $n \to \infty$.

If you try $\sum b_n = \sum \frac{1}{n^2}$ or $\sum b_n = \frac{1}{n^4}$ then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6n^2}{7n^3+3n} = 0 \leftarrow \text{not} > 0!$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{6n^4}{7n^3+3n} = \infty \leftarrow \text{not} < \infty!$$

the comparison test doesn't require that you get the $n \to \infty$ behaviour of a_n exactly - for example,

if $a_n = \frac{6}{7n^3+3n}$ and $b_n = \frac{1}{5n^2}$

then $a_n \le b_n$ for all $n \ge 5$

So $\sum_{n=1}^{\infty} \frac{6}{7n^3+3n} = a_1 + a_2 + a_3 + a_4 + \sum_{n=5}^{\infty} a_n$ and

$\sum_{n=5}^{\infty} a_n$ converges since

$a_n \leq b_n$ for all $n \geq 5$ and $\sum_{n=5}^{\infty} b_n$ converges.

ex: $\sum_{n=1}^{\infty} \frac{1}{n!}$

factorials are best done via the comparison test.

$a_1 = \frac{1}{1}$

$a_2 = \frac{1}{1}\frac{1}{2}$

$a_3 = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3}$

$a_4 = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} < \frac{1}{3} \cdot \frac{1}{4}$

$a_5 = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} < \frac{1}{4} \cdot \frac{1}{5}$

So for all $n \geq 2$ $\quad a_n \leq \frac{1}{n(n-1)} =: b_n$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n!} = a_1 + \sum_{n=2}^{\infty} a_n$ converges since $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ converges.

Note: $\sum_{2}^{\infty} b_n$ converges by the LCT and $\sum_{2}^{\infty} \frac{1}{n^2}$ converges.

Ex: $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$a_1 = \frac{1}{1}$

$a_2 = \frac{2}{2} \cdot \frac{1}{2} = \frac{1}{2}$

$a_3 = \frac{3}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} < 1 \cdot 1 \cdot 1/2$

$a_4 = \frac{4}{4} \cdot \frac{3}{4} \cdot \frac{2}{4} \cdot \frac{1}{4} < 1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2}$

$a_5 = \frac{5}{5} \cdot \frac{4}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{1}{5} < 1 \cdot 1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2}$

$a_6 = \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \cdot \frac{1}{6} < 1 \cdot 1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$a_7 = \frac{7}{7} \cdot \frac{6}{7} \cdot \frac{5}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} \cdot \frac{2}{7} \cdot \frac{1}{7} < 1 \cdot 1 \cdot 1 \cdot 1 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

$$a_n \leq \begin{cases} \left(\frac{1}{2}\right)^{n/2} & \text{if } n \text{ even} \\ \left(\frac{1}{2}\right)^{\frac{n-1}{2}} & \text{if } n \text{ odd} \end{cases} = \begin{cases} \left(\frac{1}{\sqrt{2}}\right)^{n} & \text{if } n \text{ even} \\ \left(\frac{1}{\sqrt{2}}\right)^{n-1} & \text{if } n \text{ odd} \end{cases}$$

now $\left(\frac{1}{\sqrt{2}}\right)^n = \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}\right)^{n-1} < \left(\frac{1}{\sqrt{2}}\right)^{n-1}$

So $a_n \leq \left(\frac{1}{\sqrt{2}}\right)^{n-1}$ for all $n \geq 1$

this allows us to apply the comparison test

$$a_n = \frac{n!}{n^n} < \left(\frac{1}{\sqrt{21}}\right)^{n-1} =: b_n$$

and $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{21}}\right)^{n-1}$ converges since it's a geometric series and $\left|\frac{1}{\sqrt{21}}\right| < 1$.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n!}{n^n} \text{ converges.}$$

§11.5 Alternating series

__Alternating Series Test:__

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - \dots$$

satisfies $b_n \geq 0$ for all n, $b_{n+1} \leq b_n$ for all n and $\lim_{n\to\infty} b_n = 0$ then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges.

ex: $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ converges since $b_n = \frac{1}{n} > 0$ ✓

$$b_{n+1} = \frac{1}{n+1} < \frac{1}{n} = b_n \checkmark$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0 \checkmark$$

wasn't that easy!

Also, it doesn't matter if the series is of the form $\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \dots$

since $\sum_{n=1}^{\infty} (-1)^n b_n = -b_1 + \underbrace{\sum_{n=2}^{\infty} (-1)^n b_n}$

and this has the structure that the alternating series test requires. First term is plus then minus then plus...

ex: $\sum_{n=1}^{\infty} (-1) \frac{2n}{4n^2+1}$

$b_n = \frac{2n}{4n^2+1} > 0$ ✓

$b_{n+1} < b_n$?

Let $f(x) = \frac{2x}{4x^2+1}$ then

$$f'(x) = \frac{-2(4x^2-1)}{(4x^2+1)^2} < 0$$

on $[1,\infty)$

so $b_{n+1} < b_n$ ✓

$\lim_{n\to\infty} b_n = 0$ ✓

so series converges

ex: $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ diverges. why?

$\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right) \neq 0$

and so we know
by theorem in §11.2
that the series diverges.

ex: $\sum_{n=1}^{\infty} \left(-\frac{n}{5}\right)^n$ diverges.

why? $\lim_{n\to\infty} \left(-\frac{n}{5}\right)^n \neq 0$ and so we know
from §11.2 that the series
must
diverge.

ex: $\sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n!}$

$a_1 = \frac{1}{1!}$

$a_2 = 0$

$a_3 = \frac{-1}{3!}$

$a_4 = 0$

$a_5 = \frac{1}{5!}$

$a_6 = 0$

So the series is really

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(2n-1)!}$$

and this series converges
by the alternating
series test with

$b_n = \frac{1}{(2n-1)!}$