Mat 135 Jan 7, 2005

Two more examples of the first part of the fundamental theorem of calculus.

$$\frac{d}{dx}\int_4^{3x}\sqrt{2t+1}\,dt = ?$$

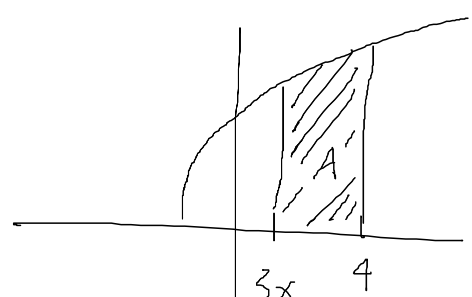
first of all, for what values of x does

$\int_4^{3x}\sqrt{2t+1}\,dt$ even make sense?

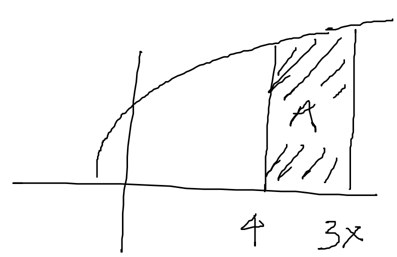
the function $f(t)=\sqrt{2t+1}$ is continuous on the domain $[-1/2,\infty)$. So we need $3x \geq -1/2$

$\Rightarrow x \geq -1/6$

if $-\frac{1}{6} \leq x < \frac{4}{3}$ then $\int_4^{3x}\sqrt{2t+1}\,dt$ is the negative of the area A ↗



if $\frac{4}{3} < x$ then $\int_4^{3x}\sqrt{2t+1}\,dt$ is the area A ↗



From the fundamental theorem of calculus, part 1 we know

$$\frac{d}{du}\int_a^u f(t)\,dt = f(u)$$

So we use this and the chain rule to compute

$$\frac{d}{dx}\int_4^{3x}\sqrt{2t+1}\,dt = \left(\frac{d}{du}\int_4^{u}\sqrt{2t+1}\,dt\right)\left(\frac{du}{dx}\right)$$ where $u = 3x$

$$= \sqrt{2u+1}\cdot 3$$

$$= \boxed{\sqrt{6x+1}\;3}$$

$$\frac{d}{dx}\int_{\cos(x)}^{\sin(x)}\frac{1}{t^2-4}\,dt$$

The function $f(t) = \frac{1}{t^2-4}$ is continuous on $(-2,2)$

So $\int_a^b f(t)\,dt$ makes sense for any a and b

such that

$[a,b]$ is in $(-2,2)$ or $[b,a]$ is in $(-2,2)$.

$\uparrow$ if $a<b$ $\uparrow$ if $b<a$.

We know $-1 \le \cos(x) \le 1$ and $-1 \le \sin(x) \le 1$ so whatever x is,

$$G(x) = \int_{\cos(x)}^{\sin(x)} \frac{1}{t^2-4}\,dt$$

makes sense. To differentiate $G(x)$ we want to use the fundamental theorem of calculus <u>and</u> the following fact:

$$\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt$$

as long as each of these definite integrals make sense

Specifically,

$$\int_{\cos(x)}^{\sin(x)} \frac{1}{t^2-4}\,dt = \int_{\cos(x)}^{0} \frac{1}{t^2-4}\,dt + \int_{0}^{\sin(x)} \frac{1}{t^2-4}\,dt$$

$$= -\int_{0}^{\cos(x)} \frac{1}{t^2-4}\,dt + \int_{0}^{\sin(x)} \frac{1}{t^2-4}\,dt$$

can use fundamental theorem of calculus and the chain rule!

$$\frac{d}{dx}\int_{\cos(x)}^{\sin(x)} \frac{1}{t^2-4}\,dt = -\frac{d}{dx}\int_{0}^{\cos(x)} \frac{1}{t^2-4}\,dt + \frac{d}{dx}\int_{0}^{\sin(x)} \frac{1}{t^2-4}\,dt$$

$$= -\left(\frac{d}{du}\int_{0}^{u} \frac{1}{t^2-4}\,dt\right)\left(\frac{du}{dx}\right) + \frac{d}{dv}\left(\int_{0}^{v} \frac{1}{t^2-4}\,dt\right)\frac{dv}{dx}$$

where $u = \cos(x)$ and $v = \sin(x)$

$$= -\left(\frac{1}{u^2-4}\right)(-\sin(x)) + \left(\frac{1}{v^2-4}\right)(\cos(x))$$

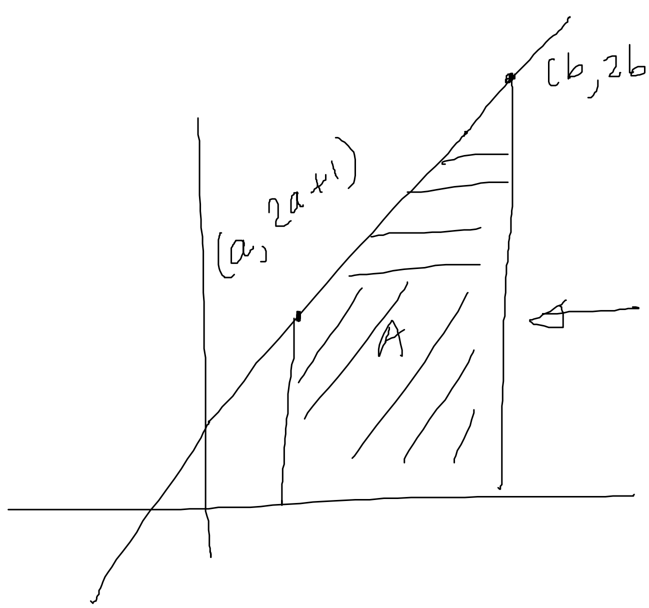
$$= \boxed{\frac{\sin(x)}{\cos^2(x)-4} + \frac{\cos(x)}{\sin^2(x)-4}}$$

On to the second part of the fundamental theorem of calculus.

Part 2: If f is continuous on $[a,b]$ and F is any antiderivative of f then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

ex: $f(x) = 2x+1$



if $b>a$ this area is $\int_a^b f(x)\,dx$

Geometrically,

$$\int_a^b 2x+1 \, dx = (2a+1)(b-a) + \frac{1}{2}\left(2b+1-(2a+1)\right)(b-a)$$
$$= b^2+b-a^2-a$$

On the other hand, if F is an antiderivative of $f(x) = 2x+1$ then $F(x) = x^2+x+C$ for some constant C.

$$F(b)-F(a) = \left[b^2+b+C\right] - \left[a^2+a+C\right]$$
$$= b^2+b-a^2-a$$

the constant C cancelled out!

Since any two antiderivatives of a function differ by a constant, we find that it doesn't matter which antiderivative we use.

Specifically, if $F'(x) = f(x)$ and $G'(x) = f(x)$ then $F(x) = G(x) + D$ for some constant D.

So

$$F(b) - F(a) = [G(b) + D] - [G(a) + D]$$
$$= G(b) - G(a).$$

translation:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

it doesn't matter <u>which</u> antiderivative of f you use.

Since it doesn't matter which antiderivative of f we use, let's see if we can find a choice that makes it clear why the second part of the fundamental theorem of calculus is true.

Let $G(x) = \int_a^x f(t)\,dt$. This is definitely an antiderivative of f. Now, $G(b) - G(a) = \int_a^b f(t)\,dt - 0$

So we've shown that for this particular antiderivative of f we have

$$G(b) - G(a) = \int_a^b f(t)\,dt$$

From before, any other antiderivative of f will also work.

<u>ex:</u>

$$\int_4^9 2x^2 + 4 - \cos(x) + \frac{1}{x}\,dx$$

$= F(9) - F(4)$ where F is an antiderivative of $2x^2 + 4 - \cos(x) + \frac{1}{x}$

$$= \frac{2}{3}x^3 + 4x - \sin(x) + \ln(x) \Big|_4^9$$

$$= \frac{2}{3}(9)^3 + 4(9) - \sin(9) + \ln(9)$$
$$- \frac{2}{3}(4)^3 - 4(4) + \sin(4) - \ln(4)$$

$$= \frac{1375}{3} + \sin(4) - \sin(9) + \ln(9) - \ln(4)$$

$\cong 457.98$ (just in case you were curious.)

ex: $\displaystyle\int_2^{10} 3x\sqrt{9x^2+1}\,dx = ?$

Whoah! What's an antiderivative of

$3x\sqrt{9x^2+1}$?!?

we know that the chain rule will give something that looks similar, if we try

$(9x^2+1)^{3/2}$ ...

$$\frac{d}{dx}(9x^2+1)^{3/2} = \frac{3}{2}(9x^2+1)^{1/2}(18x) = 27x\sqrt{9x^2+1}$$

We wanted $3x\sqrt{}$ not $27x\sqrt{}$ so we

multiply by $\frac{1}{9}$ ... $\frac{d}{dx}\frac{1}{9}(9x^2+1)^{3/2} = 3x\sqrt{9x^2+1}$ ✓

So the fundamental theorem of calculus, part 2 tells us

$$\int_2^{10} 3x\sqrt{9x^2+1}\,dx = \frac{1}{9}(9x^2+1)^{3/2}\Big|_2^{10}$$

$$= \frac{1}{9}(901)^{3/2} - \frac{1}{9}(37)^{3/2}$$

$$\cong 2979.99$$ (just in case you were curious)

And so now you see that to find

$$\int_a^b f(t)\,dt$$ you can avoid Riemann

sums and limits if.. you're good at finding antiderivatives!