Facts about sums

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} c = nc \qquad \text{(c is a number)}$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i+b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

using these, we find various properties of the definite integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\frac{b-a}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(-\frac{a-b}{n}\right)$$

$$= \lim_{n\to\infty} -\sum_{i=1}^{n} f(x_i^*)\left(\frac{a-b}{n}\right) = -\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\frac{a-b}{n}$$

$$= -\int_b^a f(x)\,dx$$

So $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

$$\int_a^a f(x) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{a-a}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\cdot 0 = \lim_{n\to\infty} 0 = 0$$

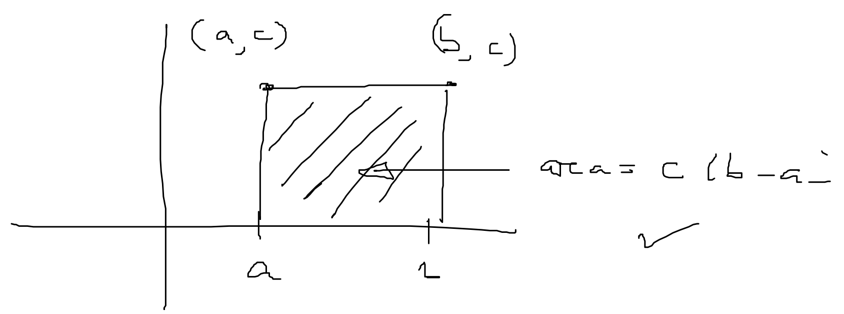
So $\displaystyle\int_a^a f(x)\,dx = 0$

$$\int_a^b c\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{b-a}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} c\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} c\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} 1$$

$$= \lim_{n\to\infty} c\frac{(b-a)}{n}\cdot n = c(b-a)$$

So $\displaystyle\int_a^b c\,dx = c(b-a)$



$$\int_a^b f(x)+g(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)+g(x_i^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x + \sum_{i=1}^{n} g(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x + \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x$$

this step is valid if each limit exists. This will be true if f and g are continuous on $[a,b]$. (Or if they have a finite number of jump or removeable discontinuities.)

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

So $$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

And so on. You can prove all sorts of things using Riemann Sums. For example, if $f(x) \le g(x)$ for all x in $[a,b]$ then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

We've used the definition of definite integral to evaluate

$$\int_a^b f(x)\,dx$$

when $f(x)$ is a polynomial or an exponential. What about for other functions? Doing it by hand turns out to be a mess. So we'll use

<u>the fundamental theorem of calculus.</u>

verrrry powerful.

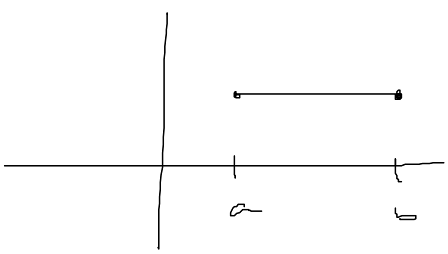
Let f be a continuous function on $[a,b]$.

define a new function, g, on $[a,b]$

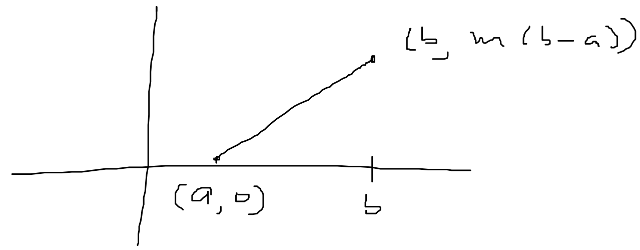
$$g(x) = \int_a^x f(t)\,dt$$

ex: $f(x) = m$ on $[a, b]$.

then $g(x) = \int_a^x m\,dt = m(x-a)$



graph of f



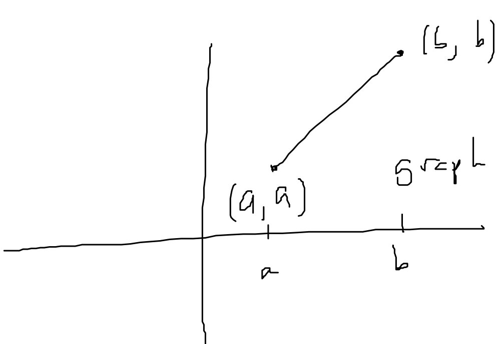
graph of g.

Note: $\frac{d}{dx} g(x) = \frac{d}{dx} \int_a^x m\,dt = m = f(x)$.

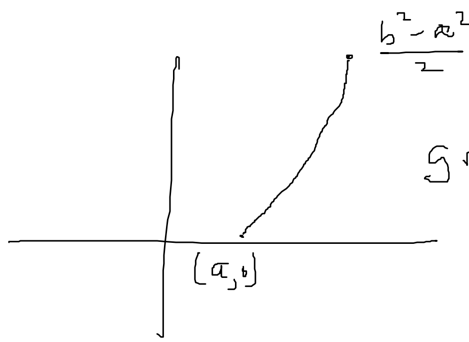
ex: $f(x) = x$ on $[a, b]$

then $g(x) = \int_a^x t\,dt = \frac{x^2}{2} - \frac{a^2}{2}$ on $[a, b]$

how? via Riemann sums or geometrically.



graph of f

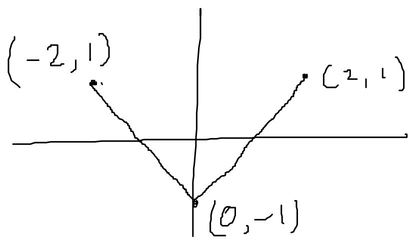


graph of g.

Note:

$$\frac{d}{dx} g(x) = \frac{d}{dx} \int_a^x t\,dt = x = f(x).$$

ex: $f(x) = |x| - 1$ on $[-2, 2]$

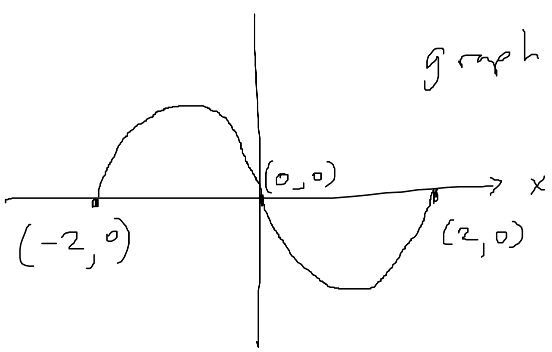


graph of f

then
$$g(x) = \int_{-2}^{x} f(t)\,dt = \begin{cases} -\frac{1}{2}x^2 - x & -2 \le x \le 0 \\ \frac{1}{2}x^2 - x & 0 < x \le 2 \end{cases}$$

how? geometrically. or via Riemann sums.

graph of g



note:
$$\frac{d}{dx} g(x) = \frac{d}{dx} \int_{-2}^{x} f(t)\,dt = \begin{cases} -x-1 & -2 < x \le 0 \\ x-1 & 0 < x < 2 \end{cases} = f(x)$$

In all three examples,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

This is the first part of the fundamental theorem of calculus.

<u>part 1</u>: if f is continuous on $[a,b]$ and $g(x) = \int_a^x f(t)\,dt$ then g is differentiable on (a,b) and $\frac{dg}{dx} = f$.

<u>ex</u>:
$$\frac{d}{dx}\int_{-\pi}^{x} \cos(t)\, e^{\sqrt{|t|}}\, \frac{\ln(t+20)}{11}\, dt$$

$$= \cos(x)\, e^{\sqrt{|x|}}\, \frac{\ln(x+20)}{11}$$

<u>ex</u>:
$$\frac{d}{dx}\int_4^{3x} \sqrt{t^3+4}\,dt = \frac{d}{du}\left(\int_4^u \sqrt{t^3+4}\,dt\right)\frac{du}{dx}$$

where $u = 3x$

So

$$\frac{d}{dx}\int_4^{3x}\sqrt{t^3+4}\,dt = \sqrt{u^3+4}\,(3) \quad \text{where } u = 3x$$

$$= \boxed{3\sqrt{27x^3+4}}$$

ex:

$$\frac{d}{dx}\int_{\cos(x)}^{5} t^4 - \sin(t)\,dt = \frac{d}{dx}\, -\int_5^{\cos(x)} t^4 - \sin(t)\,dt$$

$$= \frac{d}{du}\left(-\int_5^{u} t^4 - \sin(t)\,dt\right)\frac{du}{dx} \quad \text{where } u = \cos(x)$$

$$= -\left[u^4 - \sin(u)\right]\left(-\sin(x)\right) \quad \text{where } u = \cos(x)$$

$$= \boxed{\sin(x)\left[\cos^4(x) - \sin(\cos(x))\right]}$$

ex:

$$F(x) = \int_1^{x} f(t)\,dt \quad \text{where } f(t) = \int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du$$

find $F''(2)$

$$\frac{dF}{dx} = \frac{d}{dx}\int_1^{x} f(t)\,dt = f(x)$$

So

$$\frac{d^2F}{dx^2} = \frac{d}{dx}\frac{dF}{dx} = \frac{d}{dx} f(x)$$

$$= \frac{d}{dx}\int_1^{x^2} \frac{\sqrt{1+t^4}}{t}\,dt$$

$$= \frac{d}{du}\left(\int_1^{u} \frac{\sqrt{1+t^4}}{t}\,dt\right)\frac{du}{dx} \quad \text{where } u = x^2$$

$$= \frac{\sqrt{1+u^4}}{u}(2x) \quad \text{where } u = x^2$$

$$= \frac{\sqrt{1+x^8}}{x^2}(2x) = \frac{2\sqrt{1+x^8}}{x}$$

So $F''(2) = \dfrac{2\sqrt{1+2^8}}{2} = \boxed{\sqrt{257}}$