Jan 28, 2005 Mat 135

§7.3 Trigonometric Substitution.

We'll now learn how to deal with integrals involving

$\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, and $\sqrt{a^2+x^2}$.

This will rely on the trig identities

$$\sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2\cos^2\theta} = |a\cos\theta|$$

$$\sqrt{a^2\sec^2\theta-a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta|$$

$$\sqrt{a^2+a^2\tan^2\theta} = \sqrt{a^2\sec^2\theta} = |a\sec\theta|$$

ex: $\int x^3\sqrt{4-x^2}\,dx = ?$

this we can do by usual substitution

$u = 4-x^2 \Rightarrow du = -2x\,dx \Rightarrow \frac{-1}{2}du = x\,dx$

$x^2 = 4-u$

$$\int x^3\sqrt{4-x^2}\,dx = \int x^2\sqrt{4-x^2}\,x\,dx = -\int \frac{(4-u)\sqrt{u}}{2}\,du \quad \text{where } u = 4-x^2$$

$$= -\frac{1}{2}\int 4\sqrt{u} - u\sqrt{u}\,du \qquad \text{where } u = 4-x^2$$

$$= -\frac{1}{2}\left(\frac{8}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C \qquad \text{where } u = 4-x^2$$

$$\Rightarrow \int x^3\sqrt{4-x^2}\,dx = \boxed{-\frac{4}{3}(4-x^2)^{3/2} + \frac{1}{5}(4-x^2)^{5/2} + C}$$

What if we'd had $\int x^2\sqrt{4-x^2}\,dx$? Then we'd have been out of luck! We'd need to use a <u>different</u> substitution

$$\int x^2\sqrt{4-x^2}\,dx = ?$$

$$x = 2\sin\theta \qquad \text{where } \theta \in [-\pi/2, \pi/2]$$

$$dx = 2\cos\theta\,d\theta$$

$$\int x^2\sqrt{4-x^2}\,dx = \int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\,2\cos\theta\,d\theta$$

$$\text{where } \theta = \sin^{-1}\left(\frac{x}{2}\right) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$= \int 4\sin^2\theta\sqrt{4\cos^2\theta}\,2\cos\theta\,d\theta$$

$$\text{where } \theta = \sin^{-1}\left(\frac{x}{2}\right) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$= \int 8\sin^2\theta\cos\theta\,|2\cos\theta|\,d\theta$$

where $\theta = \sin^{-1}\left(\frac{x}{2}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$= \int 16\sin^2\theta\cos^2\theta\,d\theta \quad \text{where } \theta = \sin^{-1}\left(\frac{x}{2}\right) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

note!! we used the fact that

$\theta \in [-\pi/2, \pi/2] \Rightarrow \cos(\theta) \geq 0$

$\Rightarrow |\cos(\theta)| = \cos\theta$

$$= 16\int (\sin\theta\cos\theta)^2\,d\theta \quad \text{where } \theta = \sin^{-1}\left(\frac{x}{2}\right)$$

$$= 16\int \left(\frac{1}{2}\sin(2\theta)\right)^2 d\theta \quad \text{" "}$$

$$= 4\int \sin^2(2\theta)\,d\theta \quad \text{" "}$$

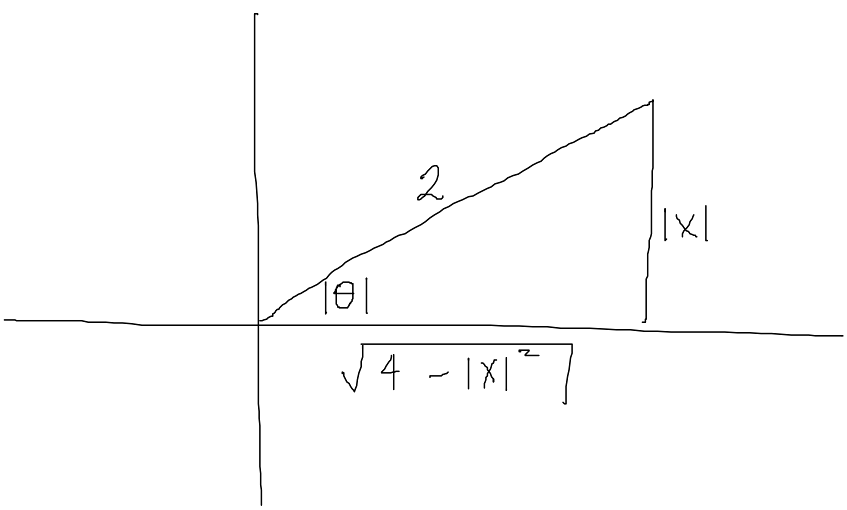
$$= 2\theta - \cos(2\theta)\sin(2\theta) \quad \text{where } \theta = \sin^{-1}\left(\frac{x}{2}\right)$$

Now we get to the tricky part. We want to write the above <u>as a function of x</u>

$$\int x^2\sqrt{4-x^2}\,dx = 2\theta - \cos(2\theta)\sin(2\theta) + C$$

where $\theta = \sin^{-1}\left(\frac{x}{2}\right)$

$\sin(\theta) = \frac{x}{2}$ and $\theta \in [-\pi/2, \pi/2]$



$\sin(\theta) = \frac{opp}{hyp}$

$$\Rightarrow \cos(\theta) = \frac{\sqrt{4-x^2}}{2}$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \left(\frac{\sqrt{4-x^2}}{2}\right)^2 - \left(\frac{x}{2}\right)^2 = 1 - \frac{x^2}{2}$$

$$\sin(2\theta) = 2\cos\theta\sin\theta = 2\left(\frac{\sqrt{4-x^2}}{2}\right)\left(\frac{x}{2}\right) = \frac{x\sqrt{4-x^2}}{2}$$

And so

$$\int x^2\sqrt{4-x^2}\,dx = 2\sin^{-1}\left(\frac{x}{2}\right) - \left(1-\frac{x^2}{2}\right)\left(\frac{x}{2}\sqrt{4-x^2}\right) + C_1$$

$$= \boxed{2\sin^{-1}\left(\frac{x}{2}\right) - \frac{x}{2}\sqrt{4-x^2} + \frac{x^3}{4}\sqrt{4-x^2} + C_1}$$

Okay, that was an example of an integral involving $\sqrt{a^2-x^2}$.

ex $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = ?$

We'll assume $a > 0$

let $x = a\sec(\theta)$

$\Rightarrow dx = a\sec(\theta)\tan(\theta)\,d\theta$

$$\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \int \frac{\sqrt{a^2\sec^2\theta - a^2}}{a^4\sec^4\theta}\,a\sec\theta\tan\theta\,d\theta$$

where $\theta = \sec^{-1}\left(\frac{x}{a}\right)$

$$= \int \frac{\sqrt{a^2\tan^2\theta}}{a^3\sec^3\theta}\tan\theta\,d\theta \quad \text{where } \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

$$= \int \frac{|a \tan\theta| \tan\theta}{a^3 \sec^3\theta} \, d\theta \quad \text{where} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

$$= \int \frac{\tan\theta \tan\theta}{a^3 \quad \sec^3\theta} \, d\theta \quad \text{where} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

note!! Here I used the fact that

$\theta = \sec^{-1}\left(\frac{x}{a}\right) \Rightarrow \theta$ w in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$

$\Rightarrow \tan(\theta) \geq 0 \Rightarrow |\tan\theta| = \tan\theta$

$$= \frac{1}{a^2} \int \frac{\tan^2\theta}{\sec^3\theta} \, d\theta \quad \text{where} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{a^2} \int \frac{\sin^2\theta}{\cos^2\theta} \cos^3\theta \, d\theta \quad \text{where} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

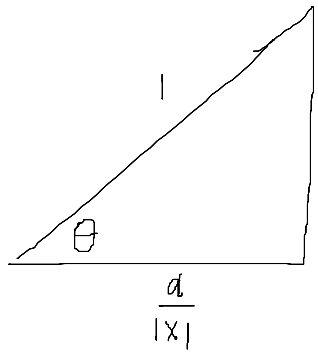
$$= \frac{1}{a^2} \frac{1}{3} \sin^3\theta + C_1 \quad \text{where} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right)$$

Now we need to write this in terms of x.

$$\sec(\theta) = \frac{x}{a}$$

$$\Rightarrow \frac{1}{\cos\theta} = \frac{x}{a} \Rightarrow \cos(\theta) = \frac{a}{x}$$

$\Rightarrow$



$$\sqrt{1-\frac{a^2}{x^2}} = \sqrt{\frac{x^2-a^2}{x^2}} = \frac{\sqrt{x^2-a^2}}{|x|}$$

recall that we assumed $a > 0$.

So $\sin\theta = \frac{\sqrt{x^2-a^2}}{x} \Rightarrow \sin^3(\theta) = \frac{(x^2-a^2)^{3/2}}{x^3}$

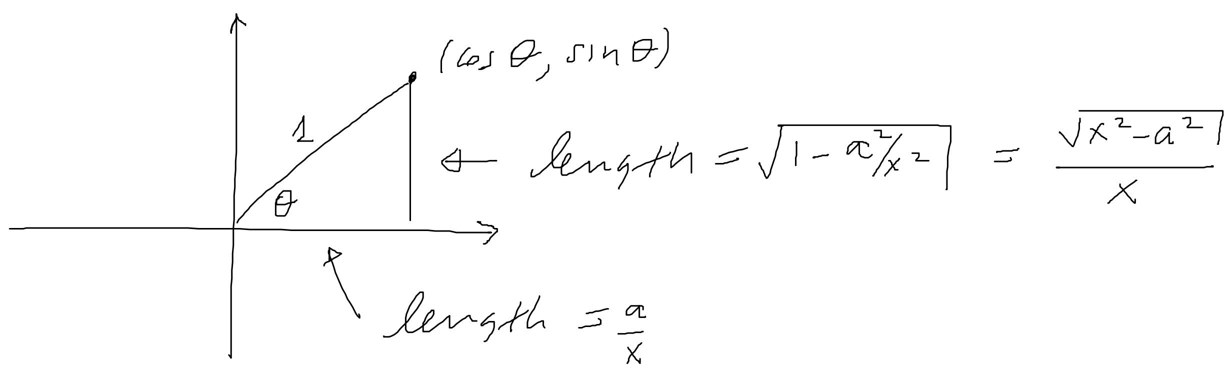
$$\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \boxed{\frac{1}{3}\frac{1}{a^2}\frac{x^3}{(x^2-a^2)^{3/2}} + C}$$

for example, $\int \frac{\sqrt{x^2-9}}{x^4}\,dx = \frac{1}{27}\frac{x^3}{(x^2-9)^{3/2}} + C$

Note: Really there are two possibilities.

Case 1: $x \geq +a > 0 \Rightarrow \sec(\theta) = \frac{x}{a} > 0$

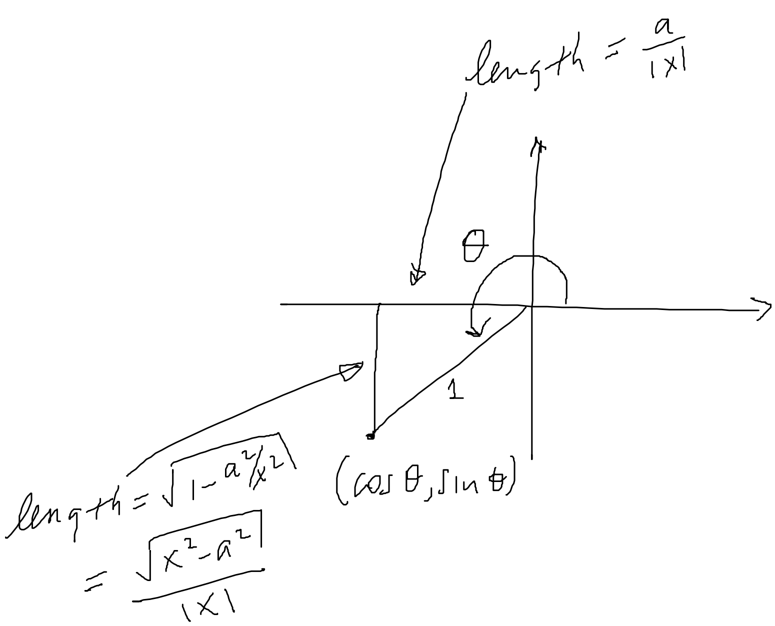
$\Rightarrow \theta$ is in $[0, \pi/2)$ in which case the triangle is



since θ is in $[0, \pi/2)$, $\sin\theta > 0 \Rightarrow \sin\theta = \frac{\sqrt{x^2-a^2}}{x}$

Case 2: $x \leq -a < 0 \Rightarrow \sec(\theta) = \frac{x}{a} < 0$

$\Rightarrow \theta$ is in $[\pi, 3\pi/2)$ in which case the triangle is



And since θ is in $[\pi, 3\pi/2)$ we know

$$\sin\theta < 0 \Rightarrow \sin\theta = \frac{\sqrt{x^2-a^2}}{x}$$

note there's no absolute value.

either way, whether $x>0$ or $x<0$,

$$\sin(\theta) = \frac{\sqrt{x^2-a^2}}{x}$$

leading to the answer given.

For integrals involving $\sqrt{a^2+x^2}$

one may need to do the substitution

$$x = a\tan\theta$$

See book.