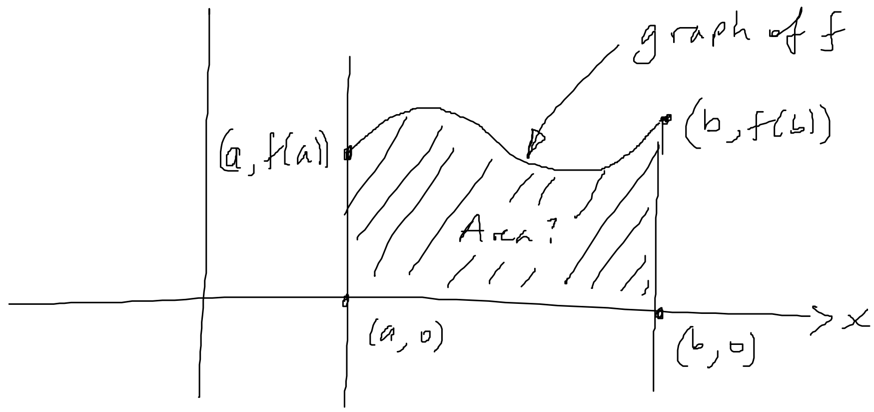
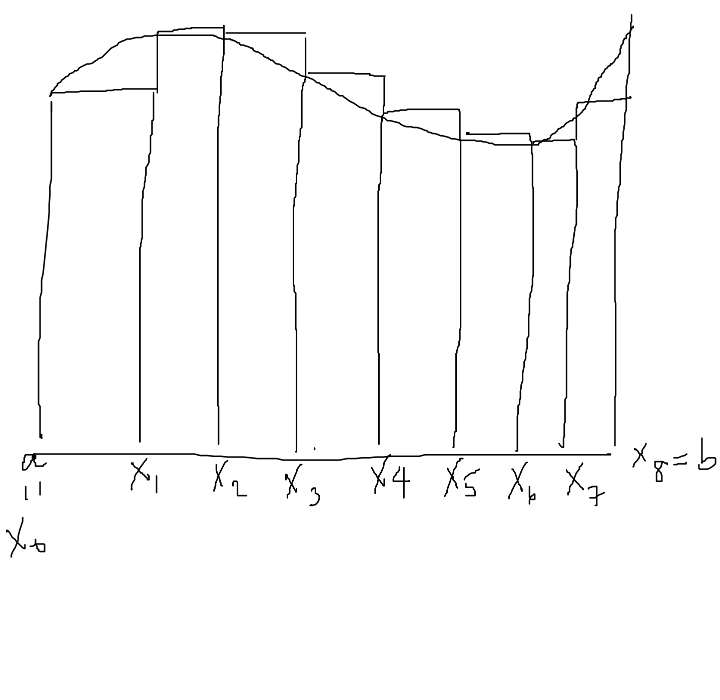
Math 135 Jan 3, 2005

From last time, when we want to find the area under the graph of f



we did this by approximating the region with thin rectangles: The rectangles have equal width (forgive the drawing!)



rectangle width

$= \Delta x = \dfrac{b-a}{\#\text{ of rectangles}}$

$= \dfrac{b-a}{8}$ (for the picture)

When we choose the height of the rectangle to be f at the left-hand endpoint of the ith rectangle (as done in the drawing) then

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

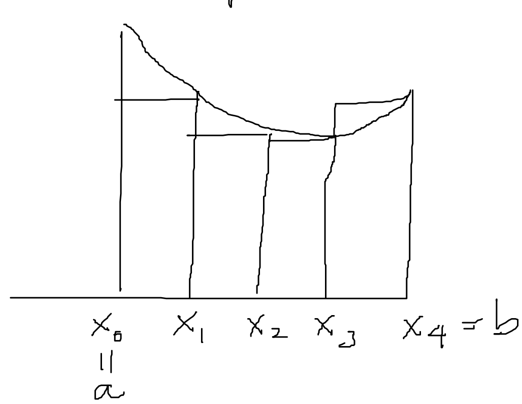
The drawing represents $L_8 = \sum_{i=0}^{7} f(x_i)\Delta x$

where $x_i = x_0 + i\,\frac{b-a}{8} = x_0 + i\Delta x$

When we choose the height of the ith rectangle to be f at the right-hand endpoint of the ith rectangle $\longrightarrow$

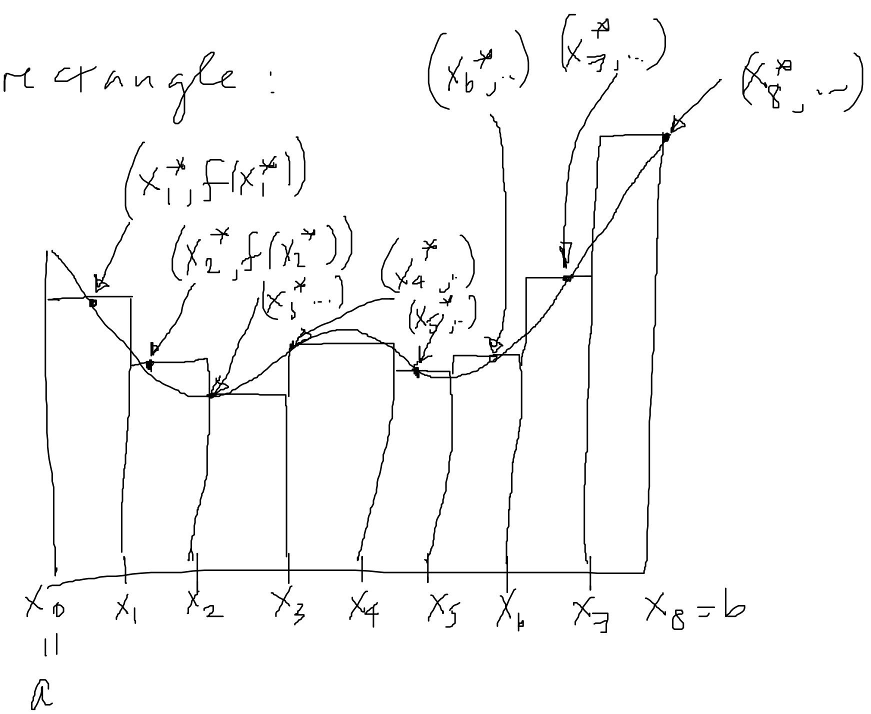
then

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$



the drawing represents $R_4 = \sum_{i=1}^{4} f(x_i)\Delta x$

In general, you could choose <u>any</u> point x_i^* in $[x_{i-1}, x_i]$ to determine the height of the i^{th} rectangle:



then $\sum_{i=1}^{8} f(x_i^*)\Delta x$ is an approximation of the area using 8 rectangles of equal width.

The choice of $x_1^*, x_2^*, \ldots x_8^*$ in the picture is different than what we'd have for the left-hand rule: $x_1^* = x_0, x_2^* = x_1, \ldots x_8^* = x_7$

or for the right-hand rule: $x_1^* = x_1, x_2^* = x_2, \ldots x_8^* = x_8$

Idea: if the function f is "nice" then however we choose the n pointers then

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

will get close to the desired area as $n \to \infty$

definition: If f is a continuous function on $[a,b]$, we divide the interval $[a,b]$ into n subintervals of equal width $\Delta x = \frac{b-a}{n}$.

We let $x_0 = a, x_1, x_2, \ldots x_n = b$ be the endpoints of the subintervals ($x_i = x_0 + i\Delta x$) and we let $x_1^*, x_2^*, \ldots x_n^*$ be any sample points in these subintervals ($x_{i-1} \leq x_i^* \leq x_i$)

then the definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

facts:

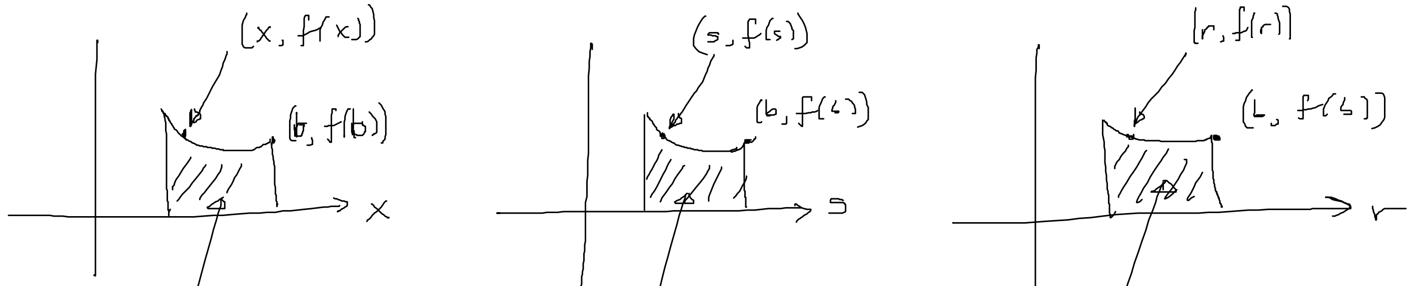
"upper limit"

$$\int_a^b \underbrace{f(x)}\,dx$$

"integrand"

"lower limit"

facts:



area = $\int_a^b f(x)\,dx$ area = $\int_a^b f(s)\,ds$ area = $\int_a^b f(r)\,dr$

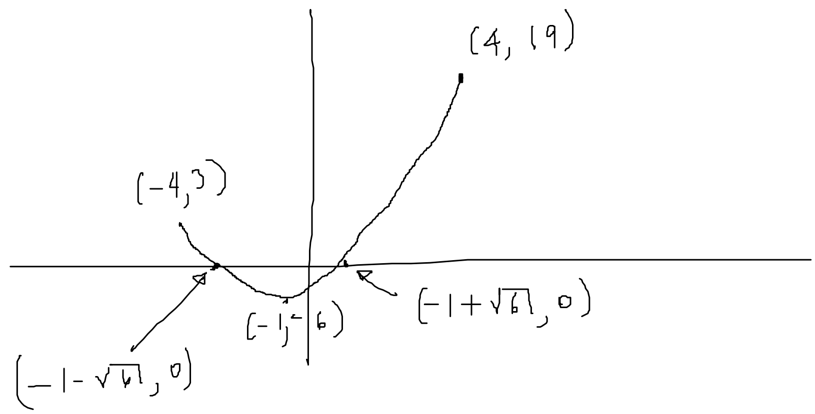
same area! so...

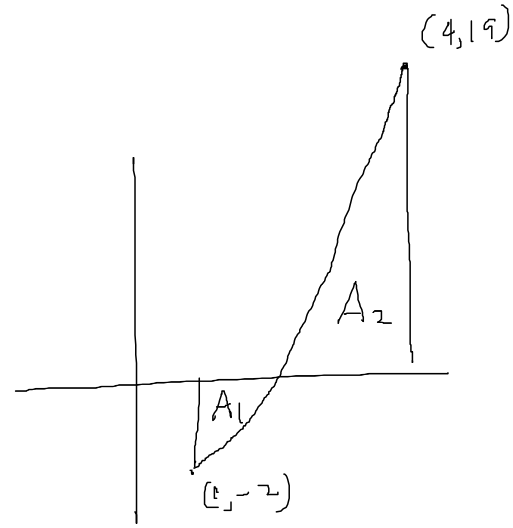
$$\int_a^b f(x)\,dx = \int_a^b f(s)\,ds = \int_a^b f(r)\,dr$$

fact: If f is continuous on $[a,b]$ then the limit in the definition of definite integral always exists and is the same number, no matter how you choose the sample points $x_1^*, x_2^*, \ldots x_n^*$. This is also true if f has a finite number of step discontinuities or removeable discontinuities.

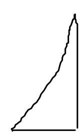
ex: evaluate the definite integral

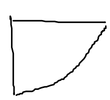
$$\int_1^4 x^2 + 2x - 5 \, dx$$





$$\int_1^4 x^2+2x-5 \, dx$$
$$= A_2 - A_1$$

where A_2 = area 

and A_1 = area of 

Since $f(x) = x^2 + 2x - 5$ is continuous on $[1, 4]$, it doesn't matter how we choose the sample points $x_1^*, x_2^*, \ldots x_n^*$. So to make my life easy, I'll choose them by the right-hand rule.

$$\int_1^4 x^2+2x-5\,dx = \lim_{n\to\infty} R_n$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{where } f(x) = x^2+2x-5$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} f(1+i\Delta x)\Delta x$$

$$= \sum_{i=1}^{n} \left[(1+i\Delta x)^2 + 2(1+i\Delta x) - 5\right]\Delta x$$

$$= \sum_{i=1}^{n} \left[1+2i\Delta x + i^2(\Delta x)^2 + 2 + 2i\Delta x - 5\right]\Delta x$$

$$= \sum_{i=1}^{n} \left[-2+4i\Delta x + i^2(\Delta x)^2\right]\Delta x$$

$$= \sum_{i=1}^{n} -2\Delta x + \sum_{i=1}^{n} 4i(\Delta x)^2 + \sum_{i=1}^{n} i^2(\Delta x)^3$$

$$= -2\Delta x \sum_{i=1}^{n} 1 + 4(\Delta x)^2 \sum_{i=1}^{n} i + (\Delta x)^3 \sum_{i=1}^{n} i^2$$

$$= -2\Delta x\left[n\right] + 4(\Delta x)^2\left[\frac{n(n+1)}{2}\right] + (\Delta x)^3\left[\frac{n(n+1)(2n+1)}{6}\right]$$

So

$$R_n = -2\left(\frac{b-a}{n}\right)[n] + 4\left(\frac{b-a}{n}\right)^2\left[\frac{n(n+1)}{2}\right] + \left(\frac{b-a}{n}\right)^3\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= -2(b-a)\left[\frac{n}{n}\right] + 2(b-a)^2\left[\frac{n(n+1)}{n^2}\right] + \frac{(b-a)^3}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right]$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left\{-2(b-a) + 2(b-a)^2\left[\frac{n(n+1)}{n^2}\right] + \frac{(b-a)^3}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right]\right.$$

$$= -2(b-a) + 2(b-a)^2 + \frac{(b-a)^3}{6}\cdot 2$$

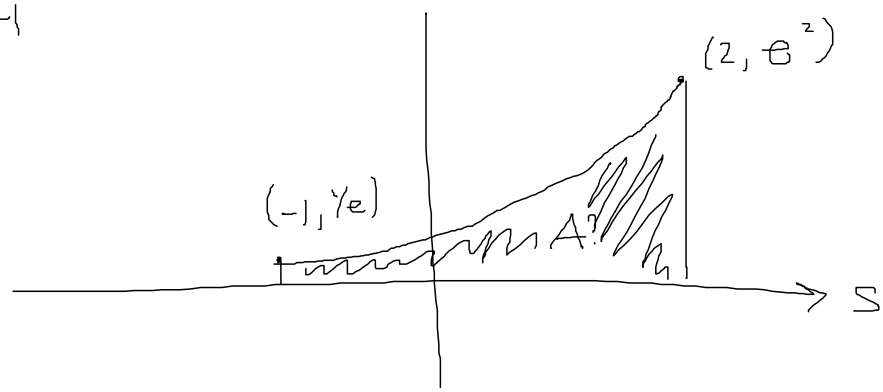
recall $b=4$, $a=1$

So

$$\int_1^4 x^2+2x-5\,dx = -2\cdot 3 + 2(3)^2 + \frac{3^3}{3}$$

$$= -6 + 18 + 9 = \boxed{21}$$

$$\int_{-1}^{2} e^{s}\,ds = ?$$



this time, I'll use the left-hand rule

$$\int_{-1}^{2} e^{s}\,ds = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=0}^{n-1} e^{x_i}\,\Delta x$$

$$R_n = \sum_{i=0}^{n-1} e^{x_i}\,\Delta x = \sum_{i=0}^{n-1} e^{-1+i\Delta x}\,\Delta x$$

$$= \sum_{i=0}^{n-1} e^{-1} e^{i\Delta x}\,\Delta x = e^{-1}\Delta x \sum_{i=0}^{n-1} e^{i\Delta x}$$

$$= e^{-1}\Delta x \sum_{i=0}^{n-1} \left(e^{\Delta x}\right)^{i}$$ recall, $\sum_{i=0}^{k} \alpha^{i} = \frac{1-\alpha^{k+1}}{1-\alpha}$

$$= e^{-1}\Delta x \, \frac{1-\left(e^{\Delta x}\right)^{n}}{1-e^{\Delta x}}$$

$$= \frac{\Delta x}{e} \, \frac{1-e^{\frac{b-a}{n}\cdot n}}{1-e^{\Delta x}}$$

so $\displaystyle\int_{-1}^{2} e^s \, ds = \lim_{n \to \infty} L_n$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{e} \frac{1 - e^3}{1 - e^{\Delta x}}$$

$$= \frac{1-e^3}{e} \lim_{\Delta x \to 0} \frac{\Delta x}{1 - e^{\Delta x}}$$

$$= \frac{1-e^3}{e} \lim_{\Delta x \to 0} \frac{1}{-e^{\Delta x}} \quad \text{(by l'Hospital's rule)}$$

$$= \frac{1-e^3}{e}(-1) = \frac{e^3 - 1}{e} = \boxed{e^2 - \frac{1}{e}}$$