Mat 135 Jan 26, 2005

In the following two weeks, we'll learn how to do all sorts of tricky integrals

Practice Practice Practice !!

§7.1 Integration by parts.

Product rule for derivatives:

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$\Rightarrow \int \frac{d}{dx}(f(x)g(x))\,dx = \int g(x)f'(x)\,dx + \int f(x)g'(x)\,dx$$

But we know $\int \frac{d}{dx}(f(x)g(x))\,dx = f(x)g(x) + C_1$!

$$\Rightarrow f(x)g(x) + C_1 = \int g(x)f'(x)\,dx + \int f(x)g'(x)\,dx$$

Of course, since the two antiderivatives on the right hand side are determined up to a constant, we can absorb the C_1 into the right hand side.

So the product rule $\Rightarrow$

$$f(x)g(x) = \int g(x)f'(x)\,dx + \int f(x)g'(x)\,dx$$

If we write this as

$$\boxed{\int f(x)\,g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx}$$

then this is the formula for integration by parts.

ex 4: $\int x e^{-x} dx = ?$

take $f(x) = x \qquad g'(x) = e^{-x}$

$\Rightarrow f'(x) = 1$ and $g(x) = -e^{-x}$

$$\Rightarrow \int x e^{-x} dx = x(-e^{-x}) - \int (-e^{-x})(1)\,dx$$

$$= -xe^{-x} + \int e^{-x} dx$$

$$\boxed{\int x e^{-x} = -xe^{-x} - e^{-x} + C}$$

ex 12: $\int p^5 \ln(p)\, dp = ?$

take $f(p) = \ln(p)$ $\quad g'(p) = p^5$

$\Rightarrow f'(p) = \frac{1}{p}$ $\quad g(p) = \frac{p^6}{6}$

$$\int p^5 \ln(p)\, dp = \frac{p^6}{6} \cdot \ln(p) - \int \frac{p^6}{6} \cdot \frac{1}{p}\, dp$$

$$= \frac{p^6}{6} \ln(p) - \frac{1}{6} \int p^5\, dp$$

$$= \boxed{\frac{p^6}{6} \ln(p) - \frac{1}{36} p^6 + C}$$

ex: 16 $\int e^{-\theta} \cos(2\theta)\, d\theta = ??$

take $f(\theta) = e^{-\theta}$ $\quad g'(\theta) = \cos(2\theta)$

$\Rightarrow f'(\theta) = -e^{-\theta}$ $\quad g(\theta) = \frac{1}{2}\sin(2\theta)$

$$\int e^{-\theta} \cos(2\theta)\, d\theta = \frac{1}{2} e^{-\theta} \sin(2\theta) + \frac{1}{2} \int e^{-\theta} \sin(2\theta)\, d\theta$$

try again!

$$f(\theta) = e^{-\theta} \qquad g'(\theta) = \sin(2\theta)$$

$$f'(\theta) = -e^{-\theta} \qquad g(\theta) = -\frac{1}{2}\cos(2\theta)$$

$\Rightarrow$

$$\int e^{-\theta}\cos(2\theta)\,d\theta = \frac{1}{2}e^{-\theta}\sin(2\theta) + \frac{1}{2}\left[-\frac{1}{2}e^{-\theta}\cos(2\theta) - \frac{1}{2}\int e^{-\theta}\cos(2\theta)\,d\theta\right]$$

$$\int e^{-\theta}\cos(2\theta)\,d\theta = \frac{1}{2}e^{-\theta}\sin(2\theta) - \frac{1}{4}e^{-\theta}\cos(2\theta) - \frac{1}{4}\int e^{-\theta}\cos(2\theta)\,d\theta$$

note! $\int e^{-\theta}\cos(2\theta)\,d\theta$ appears on both sides of the equation. So we can solve for it.

$$\frac{5}{4}\int e^{-\theta}\cos(2\theta)\,d\theta = \frac{1}{2}e^{-\theta}\sin(2\theta) - \frac{1}{4}e^{-\theta}\cos(2\theta) + C_1$$

$$\int e^{-\theta}\cos(2\theta)\,d\theta = \boxed{\frac{2}{5}e^{-\theta}\sin(2\theta) - \frac{1}{5}e^{-\theta}\cos(2\theta) + C_1}$$

Note: if $u := f(x)$ and $v := g(x)$

then $du = f'(x)dx$ and $dv = g'(x)dx$

So the Integration by parts formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

can be written as

$$\int u\,dv = uv - \int v\,du$$

Some people find this easier to remember.
It's certainly faster to write ☺

#30 $\displaystyle\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr$

$u = r^2$ $\qquad dv = \dfrac{r}{\sqrt{4+r^2}}$

$\Rightarrow du = 2r\,dr \qquad v = \sqrt{4+r^2}$

$$\int \frac{r^3}{\sqrt{4+r^2}}\,dr = r^2\sqrt{4+r^2} - 2\int r\sqrt{4+r^2}\,dr$$
$$= r^2\sqrt{4+r^2} - 2\left(\tfrac{1}{3}(4+r^2)^{3/2}\right) + C$$

So

$$\int \frac{r^3}{\sqrt{4+r^2}}\,dr = r^2\sqrt{4+r^2} - \frac{2}{3}(4+r^2)^{3/2} + C_1$$

and then

$$\int_0^1 \frac{r^3}{\sqrt{4+r^2}}\,dr = r^2\sqrt{4+r^2} - \frac{2}{3}(4+r^2)^{3/2}\Big|_0^1 = \boxed{\frac{16}{3} - \frac{7\sqrt{5}}{3}}$$

§7.2 Trigonometric Integrals

For trig integrals, we'll use trig identities to help us find the antiderivatives

Commonly used when the integrand is in terms of sin & cos:

$$(*)\quad \begin{cases} \sin^2(x) = 1 - \cos^2(x) \\ \cos^2(x) = 1 - \sin^2(x) \end{cases}$$

$$(**)\quad \begin{cases} \sin^2(x) = \frac{1}{2}(1 - \cos(2x)) \\ \cos^2(x) = \frac{1}{2}(1 + \cos(2x)) \\ \sin(x)\cos(x) = \frac{1}{2}\sin(2x) \end{cases}$$

Consider $\int \sin^{l}(x)\cos^{k}(x)\,dx$

if either l or k is odd then we'll do one thing, based on substitution, and ⊛

if both k and l are even then we'll do a different thing, based on ⊛⊛

Here's an example where at least one of l and k are odd:

#2 $\int \sin^6(x)\cos^3(x)\,dx$

$$= \int \sin^6(x)\cos^2(x)\cos(x)\,dx$$

$$= \int \sin^6(x)\left[1-\sin^2(x)\right]\cos(x)\,dx$$

$$= \int \left[\sin^6(x) - \sin^8(x)\right]\cos(x)\,dx$$

$$u = \sin(x) \qquad du = \cos(x)\,dx$$

$$= \int u^6 - u^8\,du \quad \text{where } u = \sin(x)$$

$$= \frac{u^7}{7} - \frac{u^9}{9} + C_1 \quad \text{where } u = \sin(x)$$

$$\Rightarrow \int \sin^6(x)\cos^3(x)\,dx$$

$$= \boxed{\frac{1}{7}\sin^7(x) - \frac{1}{9}\sin^9(x) + C_1}$$

\#4 $\int \cos^5(x)\,dx = ?$

$$\int \cos^5(x)\,dx = \int \cos^4(x)\cos(x)\,dx$$

$$= \int [\cos^2(x)]^2 \cos(x)\,dx$$

$$= \int [1-\sin^2(x)]^2 \cos(x)\,dx$$

$u = \sin(x) \quad du = \cos(x)\,dx$

$$= \int [1-u^2]^2\,du \quad \text{where } u=\sin(x)$$

$$= \int 1-2u^2+u^4\,du \quad \text{where } u=\sin(x)$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C_1 \quad \text{where } u=\sin(x)$$

$$\Rightarrow \int \cos^5(x)\,dx = \boxed{\sin(x) - \frac{2}{3}\sin^3(x) + \frac{1}{5}\sin^5(x) + C_1}$$

This approach will work in general. If one (or both) of the exponents is odd, then take a trig function w/ an odd exponent, set one factor aside for the substitution, convert the remaining even powers to the other trig function, and then integrate by substitution

ex: $\int \sin^9(x)\cos^7(x)\,dx$

$$= \int \sin^9(x)\cos^6(x)\ \cos(x)\,dx$$

$$= \int \sin^9(x)\left(1-\sin^2(x)\right)^3 \cos(x)\,dx$$

etc etc

or $\int \sin^9(x)\cos^7(x)\,dx = \int \sin^8(x)\cos^7(x)\ \sin(x)\,dx$

$$= \int \left(1-\cos^2(x)\right)^4 \cos^7(x)\ \sin(x)\,dx$$

etc etc

If both powers are even then we'll use the half angle formulae

$\int \cos^2(x)\,dx = ?$

$$\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$$

$$\Rightarrow \int \cos^2(x)\,dx = \frac{1}{2}\int 1 + \cos(2x)\,dx$$

$$= \frac{1}{2}x + \frac{1}{4}\sin(2x) + C_1$$

#10 $\int \cos^6(x)\,dx = ?$

$$\int \cos^6(x)\,dx = \int \left(\frac{1}{2}(1 + \cos(2x))\right)^3 dx$$

$$= \frac{1}{8}\int 1 + 3\cos(2x) + 3\cos^2(2x) + \cos^3(2x)\,dx$$

$$= \frac{1}{8}x + \frac{3}{16}\sin(2x) + \frac{3}{8}\int \cos^2(2x)\,dx + \frac{1}{8}\int \cos^3(2x)\,dx$$

using previous methods...

$$\int \cos^2(2x)\,dx = \frac{1}{2}x + \frac{1}{8}\sin(4x) + C_1$$

$$\int \cos^3(2x)\,dx = \frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C_1$$

So $\int \cos^6(x)\,dx = \frac{1}{8}x + \frac{3}{16}\sin(2x) + \frac{3}{8}\left[\frac{1}{2}x + \frac{1}{8}\sin(4x)\right]$

$$+ \frac{1}{8}\left[\frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x)\right] + C$$

$$= \boxed{\frac{5}{16}x + \frac{1}{4}\sin(2x) + \frac{3}{64}\sin(4x) - \frac{1}{48}\sin^3(2x) + C}$$

ex: $\int \sin^4(x)\cos^6(x)\,dx$

$$= \int (\sin(x)\cos(x))^4 \cos^2(x)\,dx$$

$$= \int \left(\frac{1}{2}\sin(2x)\right)^4 \left(\frac{1}{2}(1+\cos(2x))\right)dx$$

$$= \frac{1}{2^5}\int \sin^4(2x)\,(1+\cos(2x))\,dx$$

$$= \frac{1}{32}\int \sin^4(2x)\,dx + \frac{1}{32}\int \sin^4(2x)\cos(2x)\,dx$$

$$= \frac{1}{32}\int (\sin^2(2x))^2\,dx + \frac{1}{320}\sin^5(2x) \quad (\text{by subst.})$$

$$= \frac{1}{320}\sin^5(2x) + \frac{1}{32}\int \left[\frac{1}{2}(1-\cos(4x))\right]^2 dx$$

$$= \frac{1}{320}\sin^5(2x) + \frac{1}{128}\int 1 - 2\cos(4x) + \cos^2(4x)\,dx$$

$$= \frac{1}{320}\sin^5(2x) + \frac{x}{128} - \frac{1}{256}\sin(4x) + \frac{1}{128}\int \cos^2(4x)\,dx$$

$$= \frac{1}{320}\sin^5(2x) + \frac{x}{128} - \frac{1}{256}\sin(4x) + \frac{1}{128}\int \frac{1}{2}\left(1+\cos(8x)\right)dx$$

$$= \frac{1}{320}\sin^5(2x) + \frac{x}{128} - \frac{1}{256}\sin(4x) + \frac{x}{256} + \frac{\sin(8x)}{2048} + C_1$$

$$= \boxed{\frac{3}{256}x + \frac{1}{320}\sin^5(2x) - \frac{1}{256}\sin(4x) + \frac{1}{2048}\sin(8x) + C_1}$$

ex:

$$\int \sin(x)\cos(3x)\,dx$$

$$= \int \sin(x)\left[\cos(2x)\cos(x) - \sin(2x)\sin(x)\right]dx$$

$$= \int \sin(x)\cos(x)\cos(2x)\,dx - \int \sin^2(x)\sin(2x)\,dx$$

$$= \int \sin(x)\cos(x)\left[\cos^2(x) - \sin^2(x)\right] dx$$
$$- \int \sin^2(x)\, 2\sin(x)\cos(x)\, dx$$

$$= \int \sin(x)\cos^3(x)\, dx - \int \sin^3(x)\cos(x)\, dx$$
$$-2\int \sin^3(x)\cos(x)\, dx$$

$$= \boxed{-\frac{1}{4}\cos^4(x) - \frac{3}{4}\sin^4(x) + C_1}$$

Alternately, if you're good at memorizing formulae, you can use

$$\sin(A)\cos(B) = \tfrac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$$

$$\sin(A)\sin(B) = \tfrac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos(A)\cos(B) = \tfrac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

this implies

$$\sin(x)\cos(3x) = \frac{1}{2}\left[\sin(x-3x) + \sin(x+3x)\right]$$
$$= \frac{1}{2}\left[\sin(-2x) + \sin(4x)\right]$$
$$= \frac{1}{2}\left[-\sin(2x) + \sin(4x)\right]$$

$$\Rightarrow \int \sin(x)\cos(3x)\,dx = \frac{1}{2}\int -\sin(2x) + \sin(4x)\,dx$$

$$= \frac{1}{2}\left(\frac{\cos(2x)}{2} - \frac{\cos(4x)}{4}\right) + C$$

$$= \boxed{\frac{1}{4}\cos(2x) - \frac{1}{8}\cos(4x) + C}$$

Note: $\frac{1}{4}\cos(2x) - \frac{1}{8}\cos(4x) = -\frac{1}{4}\cos^4(x) - \frac{3}{4}\sin^4(x) - 3/8$

so the two methods agree.

Similarly, you can do trig integrals involving $\sec(x)$ and $\tan(x)$ by taking advantage of the trig identities involving them. See book.