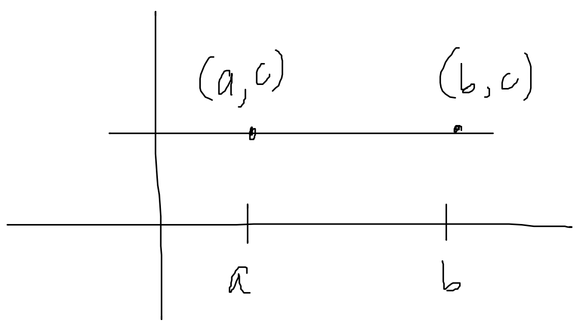
Mat 135 Jan 24, 2005

§6.5 Average Value of a function.

Consider $f(x) = c$ on $[a,b]$

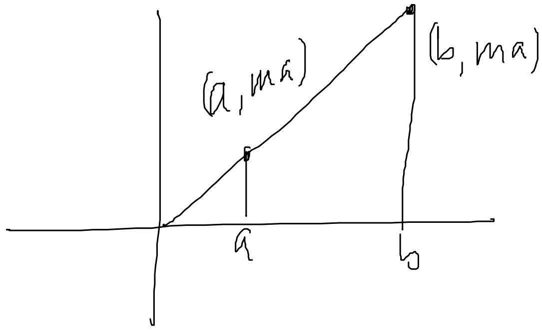


What is your sense of "the average value of f on $[a,b]$"? Probably c.

<u>note:</u> $$\int_a^b f(x)\,dx = c\int_a^b dx = c(b-a)$$

$$\Rightarrow \frac{1}{b-a}\int_a^b f(x)\,dx = c$$

How about $f(x) = mx$ on $[a,b]$?

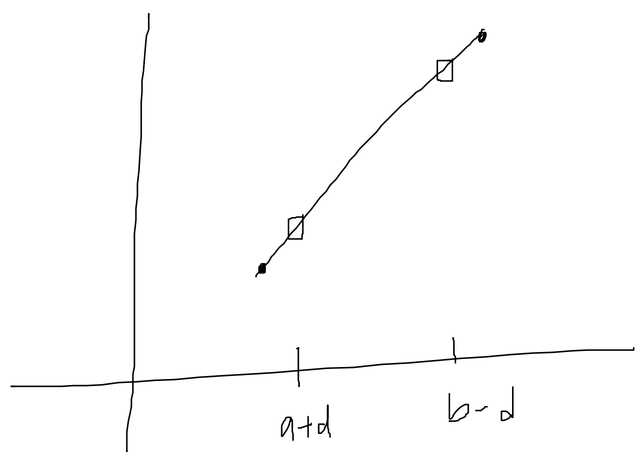


In this case, the average of

$f(a)$ and $f(b)$ is $\frac{1}{2}ma + \frac{1}{2}mb$

so the average of f at the endpoints is $\frac{1}{2}m(a+b)$

If we move in from the endpoints by a distance d and ask for the average ..

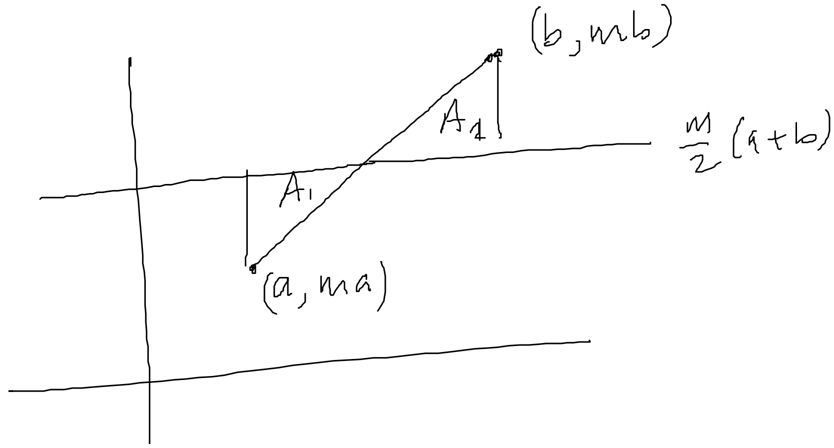


Then the average is $\frac{(f(a+d) + f(b-d))}{2}$

$$= \frac{ma + md + (mb - md)}{2}$$

$$= \frac{1}{2}m(a+b)$$

guess! average $= \frac{m}{2}(a+b)$



if we have the right average, the area above the average value will equal the area below the average value. That is

$$\int_a^b (f(x) - f_{av})\,dx = 0$$

$$\Rightarrow \int_a^b f(x)\,dx - \int_a^b f_{av}\,dx = 0$$

$$\Rightarrow \int_a^b f(x) - f_{av}(b-a) = 0$$

$$\Rightarrow f_{av} = \frac{\int_a^b f(x)\,dx}{b-a}$$

Let's check this for the straight line case

$$f_{av} = \frac{\int_a^b mx \, dx}{b-a}$$

$$= \frac{m \frac{x^2}{2} \Big|_a^b}{b-a}$$

$$= \frac{\frac{m}{2}(b^2-a^2)}{b-a} = \frac{m}{2}(b+a) \checkmark$$

Note: My idea of going in a distance d and averaging $f(a+d)$ and $f(b-d)$ is okay for linear functions but doesn't always work.

The important thing about averages is the observation that $\int_a^b f(x) - f_{av} \, dx = 0$ that is, the area above f_{av} equals the area below f_{av}.

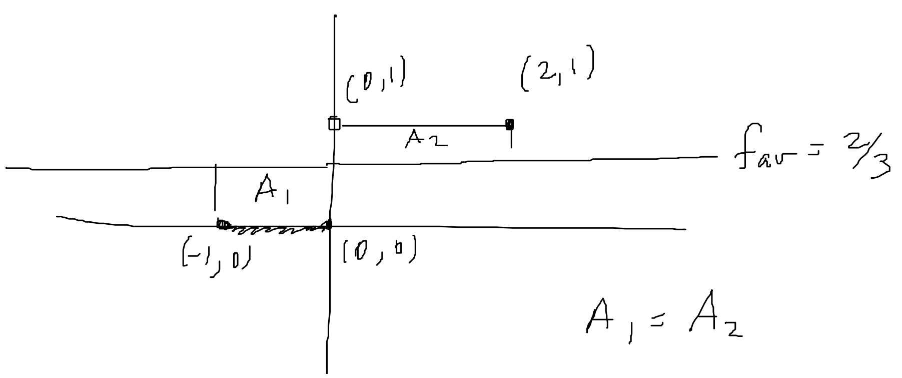
See book for alternate derivation of $f_{av} = \frac{\int_a^b f(x)\,dx}{b-a}$

Q: Is there always a point c so that $f(c) = f_{av}$? i.e. is the average value always achieved?

A: Not necessarily. Consider

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases}$$

on $[-1, 2]$. Then $f_{av} = 2/3$ which isn't achieved.



Fact! If f is continuous on $[a,b]$ then f_{av} is achieved at some point c.

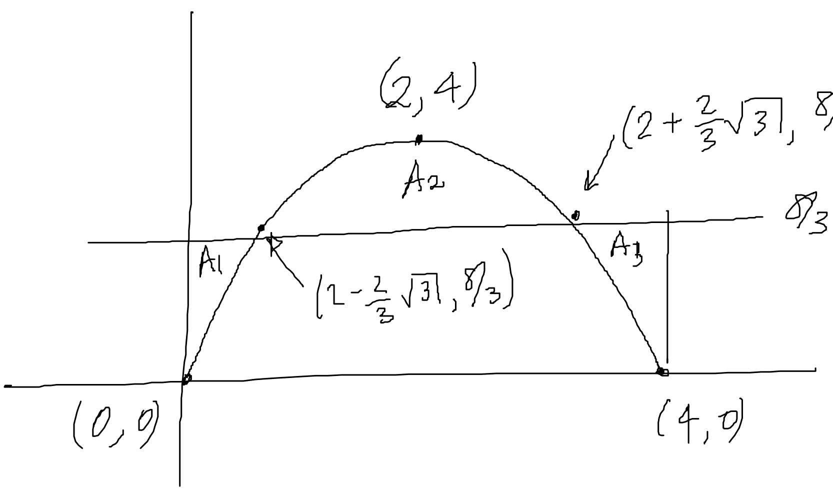
MVT for integrals: If f is continuous on $[a,b]$ then there is a $c \in [a,b]$ so that

$$f(c) = f_{av} \Rightarrow f(c)(b-a) = \int_a^b f(x)\,dx$$

<u>Note:</u> there could be more than one point at which f_{av} is achieved!

<u>ex:</u> consider $f(x) = 4x - x^2$ on $[0, 4]$

$$f_{av} = \frac{\int_0^4 4x - x^2\,dx}{4 - 0} = 8/3$$



$$A_1 + A_3 \stackrel{?}{=} A_2$$

$$\int_0^{2-\frac{2}{3}\sqrt{3}} \frac{8}{3} - (4x - x^2)\,dx + \int_{2+\frac{2}{3}\sqrt{3}}^{4} \frac{8}{3} - (4x - x^2)\,dx \stackrel{?}{=} \int_{2-\frac{2}{3}\sqrt{3}}^{2+\frac{2}{3}\sqrt{3}} (4x - x^2) - \frac{8}{3}\,dx$$

$$\frac{16}{27}\sqrt{3} + \frac{16}{27}\sqrt{3} \stackrel{?}{=} \frac{32}{27}\sqrt{3}$$ true!

ex 10 a) find f_{av}

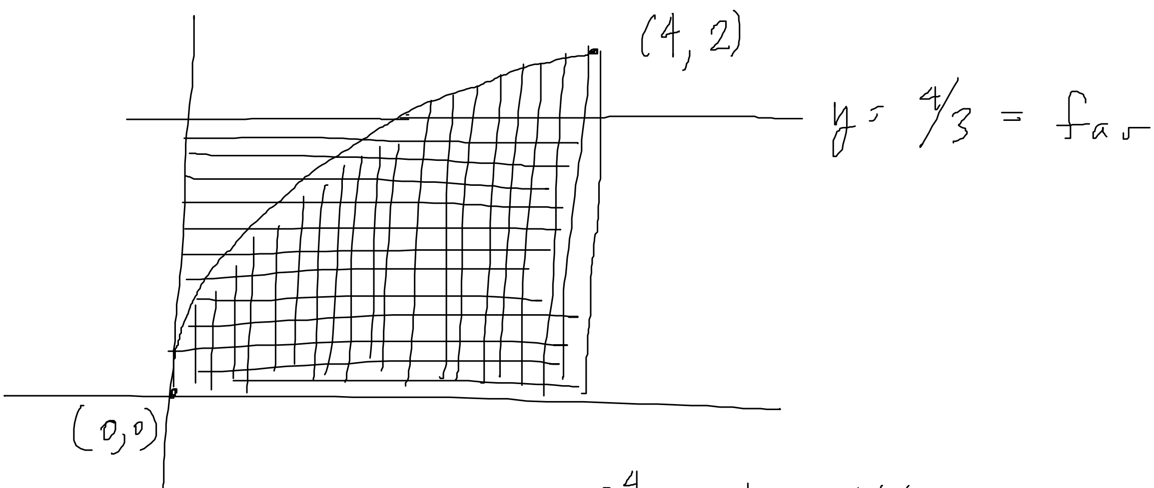
b) find c such that $f(c) = f_{av}$

c) sketch graph and a rectangle whose area is the same as the area under the graph of f.

$f(x) = \sqrt{x}$ on $[0, 4]$

$$f_{av} = \frac{\int_0^4 \sqrt{x}\,dx}{4-0} = \frac{\frac{2}{3} x^{3/2} \Big|_0^4}{4-0} = \boxed{\frac{4}{3}}$$

$$f(c) = 4/3 \Rightarrow \sqrt{c} = 4/3 \Rightarrow c = 16/9 \cong 1.78$$



||| area $= \int_0^4 \sqrt{x}\, dx = 16/3$

$\equiv$ area $= \frac{4}{3} \cdot (4-0) = 16/3$

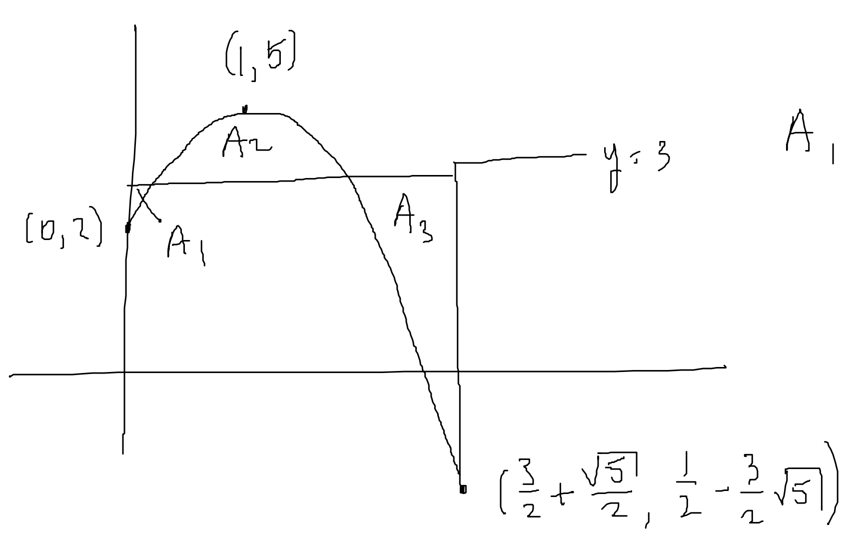
ex 14 find b such that the average value of $f(x) = 2 + 6x - 3x^2$ on $[a,b]$ equals 3.

$$3 = \frac{\int_0^b 2 + 6x - 3x^2 \, dx}{b - 0}$$

$$3 = \frac{2x + 3x^2 - x^3 \Big|_0^b}{b - 0} = \frac{2b + 3b^2 - b^3}{b}$$

$$3 = 2 + 3b - b^2 \Rightarrow b = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$$

Case 1: $b = \frac{3}{2} + \frac{\sqrt{5}}{2} \cong 2.62$

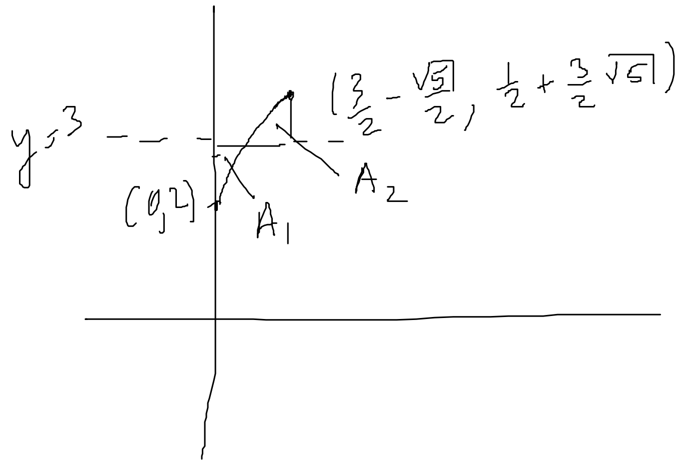


$A_1 + A_3 = A_2$

(okay, the graph isn't convincing...)

case 2: $b = \frac{3}{2} - \frac{\sqrt{5}}{2} \cong 0.38$ $\qquad f(b) = \frac{1}{3} + \frac{3}{2}\sqrt{5}$

$\cong 3.85$



$A_1 = A_2$

ex 18 If a cup of coffee has temperature 95°C in a 20°C room, then according to Newton's law of cooling, the temperature of the coffee after t minutes is

$$T(t) = 20 + 75e^{-t/50}$$

What is the average temperature of the coffee in the first half hour?

$T(0) = 20 + 75e^{0} = 20 + 75 = 95$ ✓

as $t \to \infty$ $\quad T(t) \to 20$ ✓

$$T_{av} = \frac{\int_0^{30} T(t)\,dt}{30-0}$$

$$= \frac{1}{30}\int_0^{30} 20 + 75e^{-t/50}\,dt$$

$$= \frac{1}{30}\left(20t - 3750\, e^{-t/50} \Big|_0^{30} \right)$$

$$= 145 - 125e^{-3/5} \cong 76.40$$

$T(30) \cong 61.16$