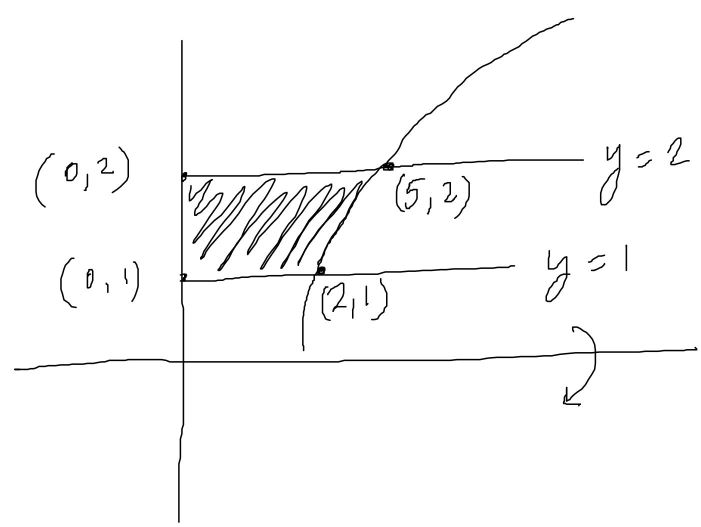
§6.3 Volume by the method of Shells.

Rather than slicing the solid of rotation into disks/washers, we will now consider building the solid out of nested shells.

ex: Consider the region bounded by

$x = 1+y^2$, $x=0$, $y=1$, $y=2$

rotated about the x-axis. Find its volume.



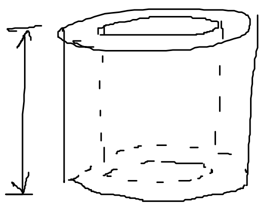
by the method of washers, we'll get a dx integral

$$\int \pi \left(r_{out}(x)^2 - r_{in}(x)^2\right) dx$$

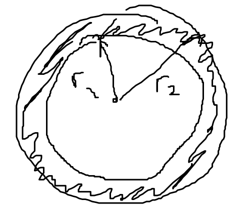
$$\text{volume} = \int_0^2 \pi\left(2^2-1^2\right)dx + \int_2^5 \pi\left(2^2-\left(\sqrt{x-1}\right)^2\right)dx = \boxed{\frac{21\pi}{2}}$$

In this calculation, we constructed the solid object out of "washers" of height Δx.

Now we'll do it with nested shells.



seen from above:
r_1 = inner radius
r_2 = outer radius



$$\text{volume of shell} = h\pi r_2^2 - h\pi r_1^2$$

$$= h\pi\left(r_2^2-r_1^2\right)$$

$$= h\pi\left(r_1+r_2\right)\left(r_2-r_1\right)$$

$$= 2\pi\left(\frac{r_1+r_2}{2}\right) h\left(r_2-r_1\right)$$

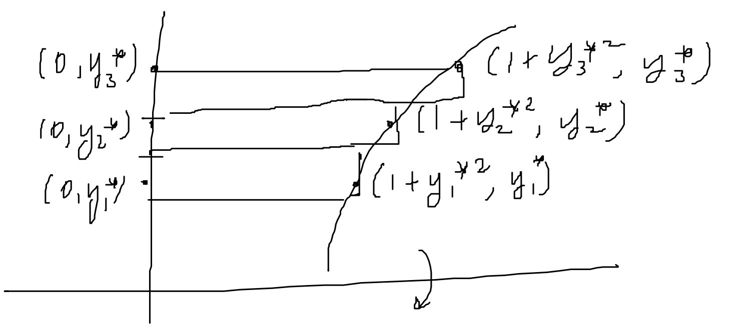
$$\frac{r_1+r_2}{2} =: r = \text{average radius}$$

$$r_2-r_1 =: \Delta r = \text{thickness of shell}$$

So the volume of the shell is

$2\pi r \cdot h \cdot \Delta r$ where r = average radius

Vol = circumference · height · thickness.



If you rotate these 3 rectangles about the x-axis you'll find three shells. The volume of the three shells will approximate the desired volume

inner shell $V \cong 2\pi y_1^* (1+y_1^{*2})\Delta y$

middle shell $V \cong 2\pi y_2^* (1+y_2^{*2})\Delta y$

outer shell $V \cong 2\pi y_3^* (1+y_3^{*2})\Delta y$

So

desired volume $\cong \sum_{i=1}^{3} 2\pi y_i^* (1 + y_i^{*2}) \Delta y$

for n slices,

desired volume $\cong \sum_{i=1}^{n} 2\pi y_i^* (1 + y_i^{*2}) \Delta y$

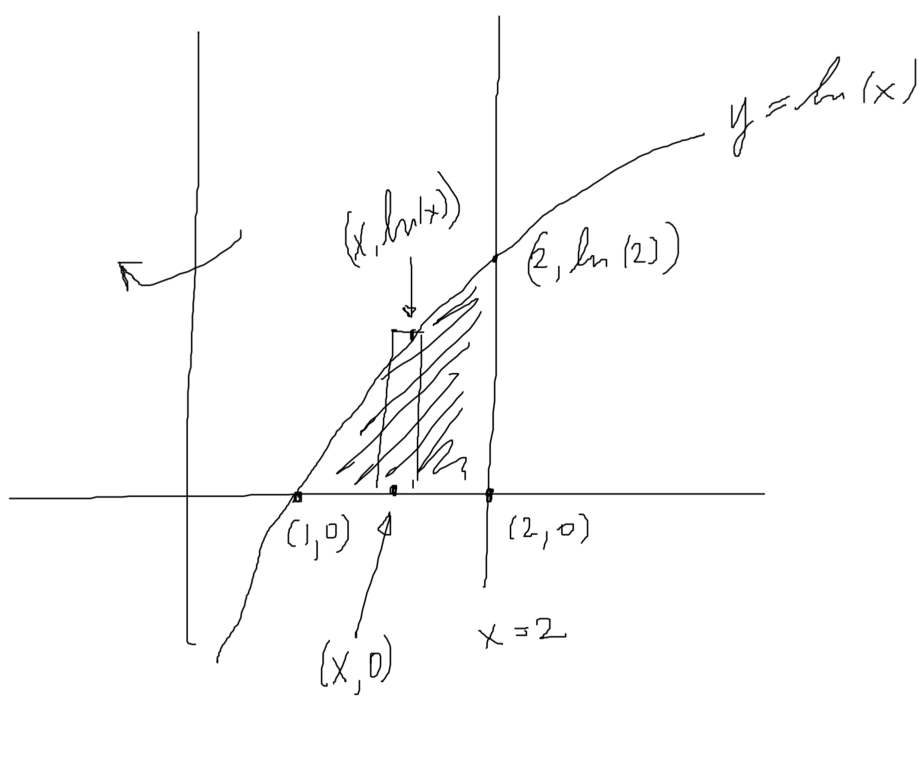
so as $n \to \infty$

$$V = \int_1^2 2\pi y (1 + y^2)\, dy = \boxed{\frac{21\pi}{2}}$$ ✓

ex 21: rotate the region bounded by

$y = \ln(x)$, $y = 0$, $x = 2$

about the y-axis. What is the volume of this solid?



rotating that tall thin rectangle about the y axis, you find a shell of volume

$$2\pi x\,(\ln(x) - 0)\,\Delta x$$

$$= 2\pi x \ln(x)\,\Delta x$$

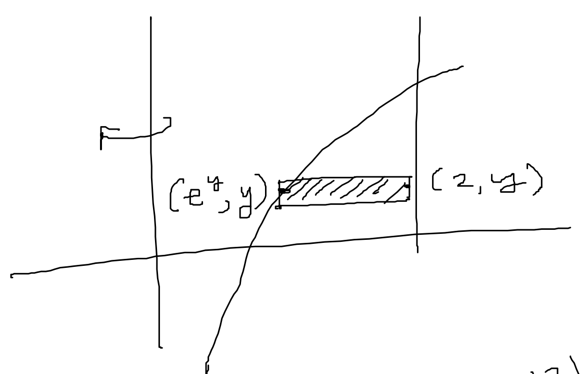
$\Rightarrow$ volume $= \displaystyle\int_1^2 2\pi x \ln(x)\,dx = \boxed{4\pi \ln(2) - \frac{3\pi}{2}}$

and here I used

$$\int 2\pi x \ln|x| = \pi x^2 \ln(x) - \frac{\pi}{2} x^2$$

this follows from integration by parts, which you'll learn soon.

Note: if you'd done this by the method of washers...



washer volume $= \pi\left(2^2 - (e^y)^2\right)\Delta y$

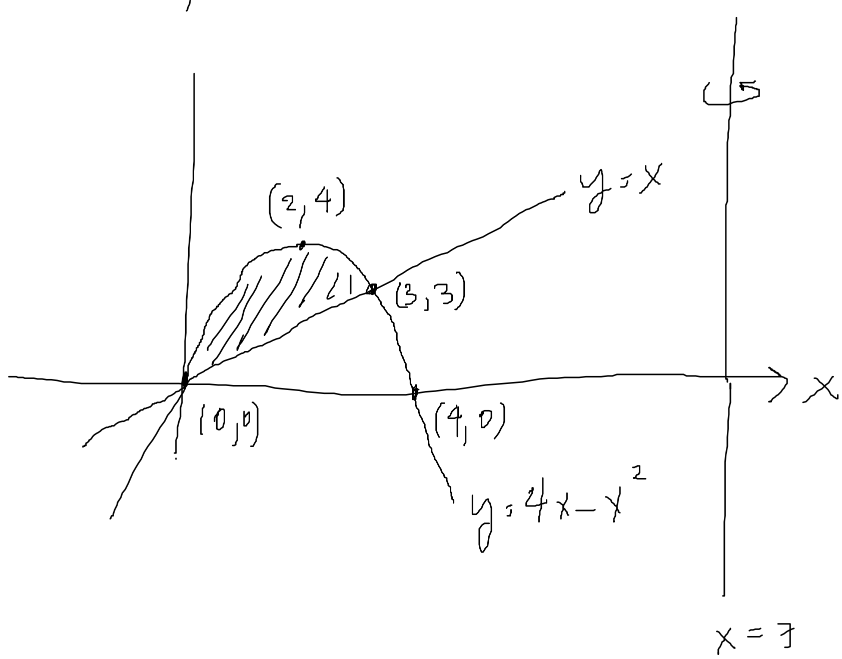
and so

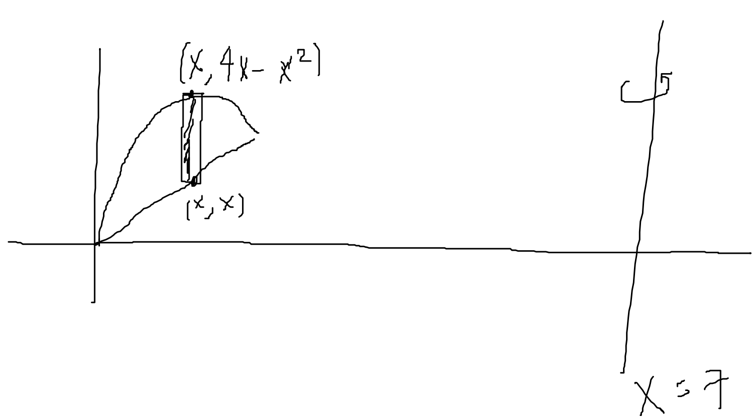
$$\text{vol} = \int_0^{\ln(2)} \pi\left(4 - e^{2y}\right) dy = \boxed{4\pi\ln(2) - \frac{3\pi}{2}}$$ ✓

ex #22 Rotate the region bounded by

$$y = x \qquad y = 4x - x^2$$

about $x = 7$. What is the volume of the resulting solid?





rotate rectangle about $x = 7$ to find a shell the volume of the shell is

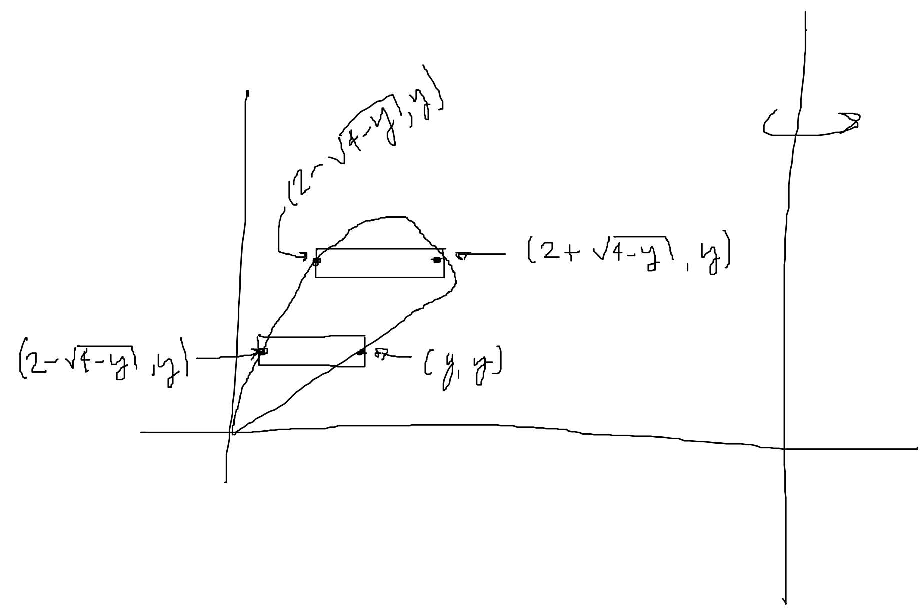
$$2\pi(7-x)\left[(4x-x^2)-x\right]\Delta x$$

$$= 2\pi(7-x)(3x-x^2)\Delta x$$

So the volume of the object is

$$\int_0^3 2\pi(7-x)(3x-x^2)\,dx = \boxed{\frac{99}{2}\pi}$$

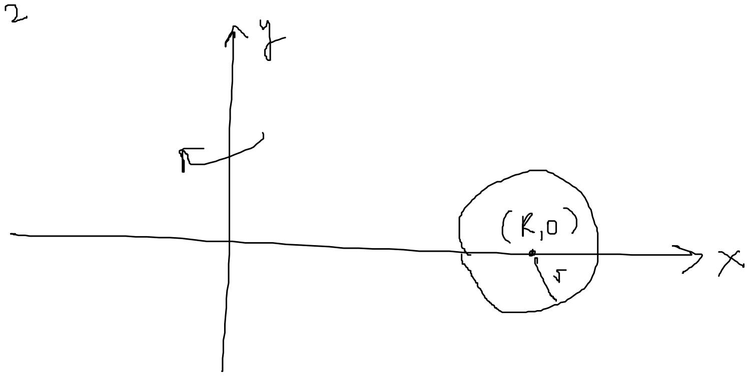
If for some reason you'd done this calculation by the method of discs...



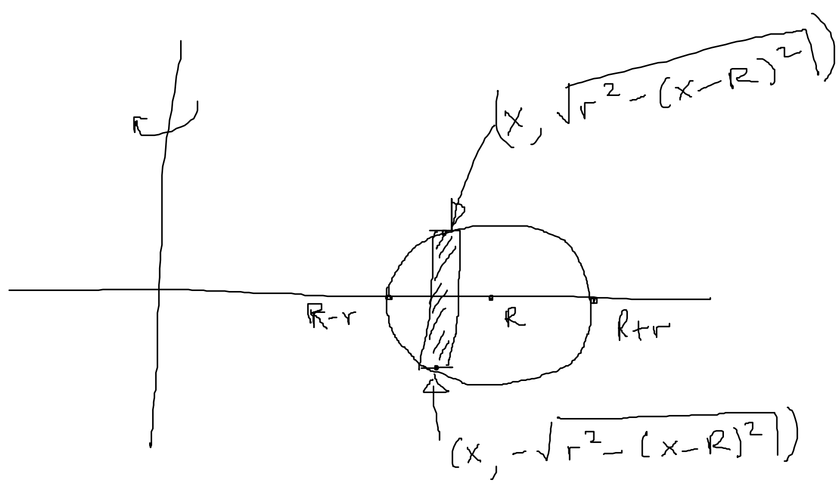
$$\int_0^3 \pi\left(\left(7-(2-\sqrt{4-y})\right)^2-(7-y)^2\right)dy + \pi\int_3^4 \left(7-(2-\sqrt{4-y})\right)^2-\left(7-(2+\sqrt{4-y})\right)^2 dy$$

$$= \boxed{\frac{99}{2}\pi}$$

We now return to the torus. By the method of washers, we know its volume is $2\pi^2 R r^2$



now by the method of shells.



rotate the rectangle and you'll get a shell of volume

$$2\pi x \left(\sqrt{r^2-(x-R)^2} - \left(-\sqrt{r^2-(x-R)^2}\right)\right) \Delta x$$

$$= 4\pi x \sqrt{r^2-(x-R)^2}\, \Delta x$$

So the volume is

$$\int_{R-r}^{R+r} 4\pi x\sqrt{r^2-(x-R)^2}\,dx = \boxed{2\pi^2 R r^2}$$

Above, I used

$$\int x\sqrt{r^2-(x-R)^2}\,dx = \int (u+R)\sqrt{r^2-u^2}\,du \quad \text{where } u=x-R$$

$$= \int u\sqrt{r^2-u^2}\,du + R\int\sqrt{r^2-u^2}\,du \quad \text{where } u=x-R$$

$$= -\frac{1}{3}(r^2-u^2)^{3/2} + R\left[\frac{u}{2}\sqrt{r^2-u^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{u}{r}\right)\right] + C \quad \text{where } u=x+R$$

(Here, I used a trig substitution to do the second antiderivative. You'll learn how to do this soon.)

$$= -\frac{1}{3}\left(r^2-(x-R)^2\right)^{3/2} + R\left[\frac{(x-R)}{2}\sqrt{r^2-(x-R)^2} + \frac{r^2}{2}\sin^{-1}\left(\frac{x-R}{r}\right)\right] + C$$