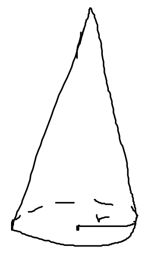
Mat 135, Jan 19 2005

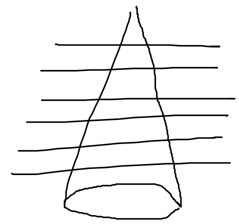
§6.2 Volumes of solids.

Given a cone



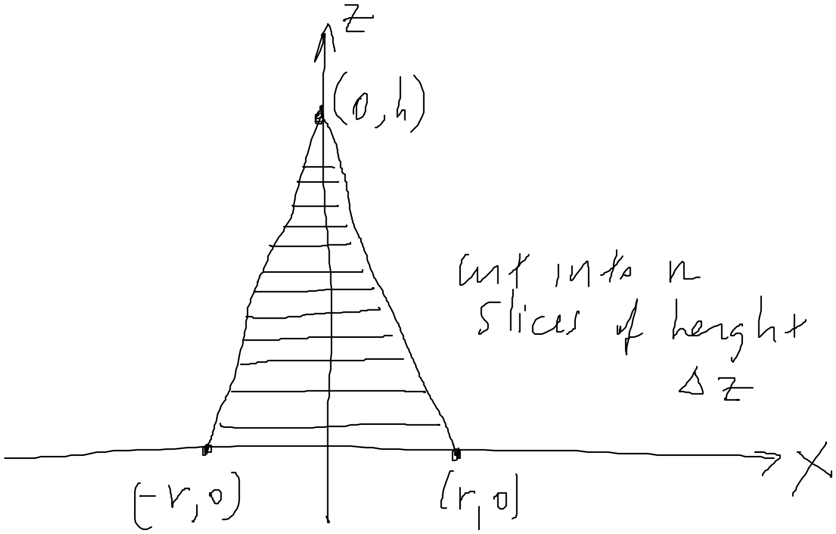
of height h and base radius r then its volume is $\frac{1}{3}\pi r^2 h$ why?

Cut the cone into slices

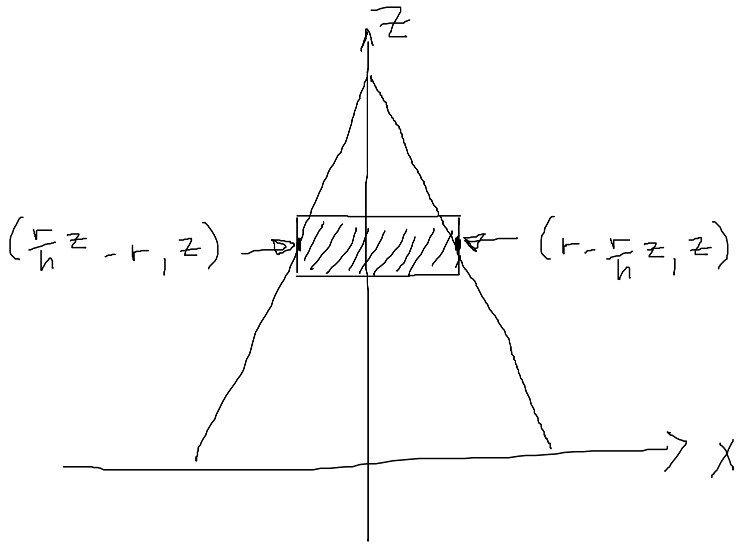


each slice will be approximately a disc.

We know the volume of a disc, so we'll approximate the volumes of the slices with the volumes of the discs



← the cone when intersected with the xz plane



that's the approximating disc, intersected with the xz plane

disc volume is

$$\pi \left(r - \frac{r}{h} z\right)^2 \Delta z$$

since its radius is

$$r - \frac{r}{h} z$$

Note: the radius of the approximating disc depends on z! Let's check we've done it right: $z = h \Rightarrow$ radius better $= 0$ true!

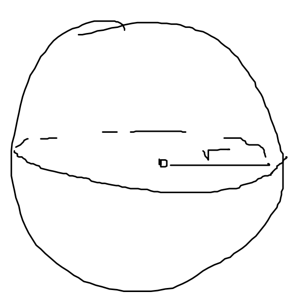
$z = 0 \Rightarrow$ radius better $= r$ true!

So the volume of the cone, when approximated by n disks of equal height is approximately

$$\sum_{i=1}^{n} \pi \left(r - \frac{r}{h} z_i^*\right)^2 \Delta z$$

taking $n \to \infty$ we get $\int_0^h \pi\left(r - \frac{r}{h} z\right)^2 dz = \boxed{\frac{1}{3}\pi r^2 h}$

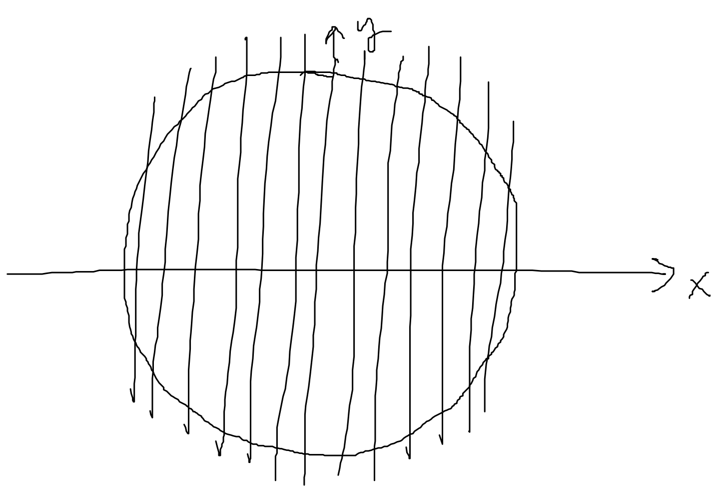
How about the volume of a sphere?



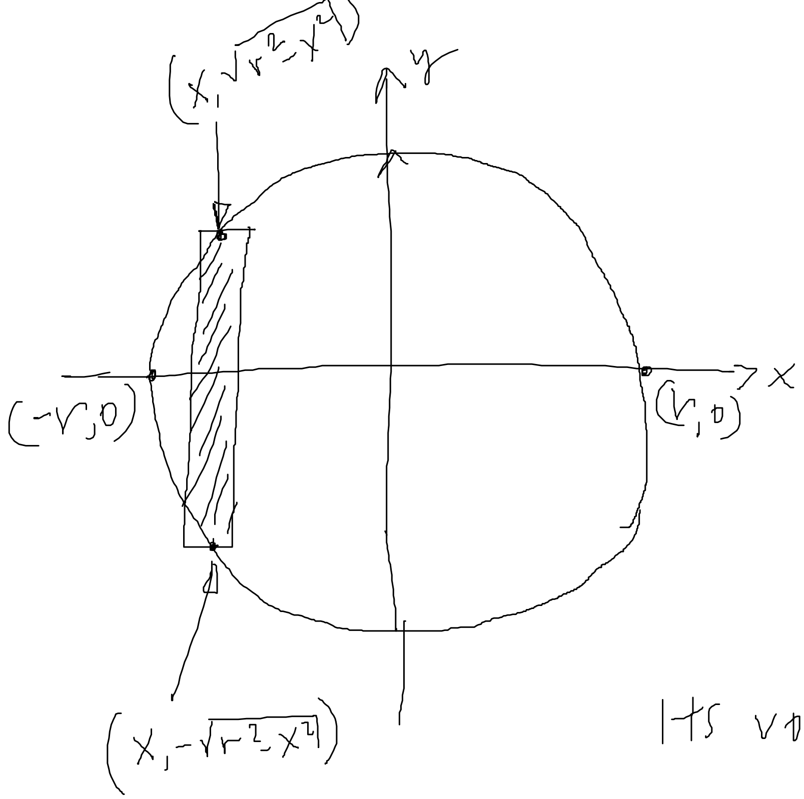
we've been told

$$V = \frac{4}{3}\pi r^2$$

again, we'll cut it into n slices



← sphere intersected with xy plane, cut into n equal-width slices



we approximate the slices with discs. here's one of the discs, intersected with the xy plane

Its volume is

$$\pi\left(\sqrt{r^2-x^2}\right)^2 \Delta x$$

note that the ~~volume~~ radius of the disc depends on x. It'd better equal 0 when $x=-r$ (true!) and when $x=r$ (true!) or we've made a mistake.

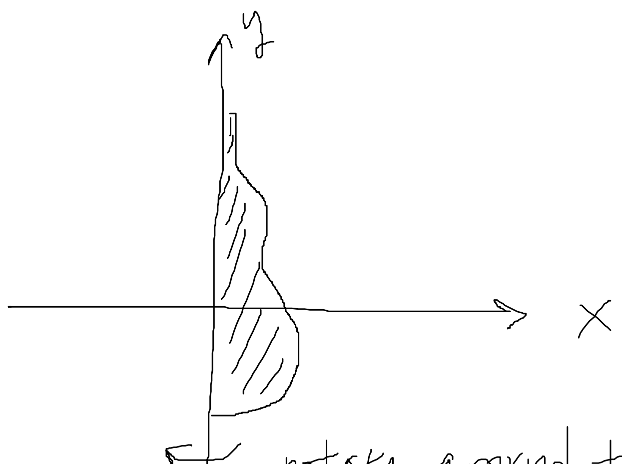
So volume of sphere $\cong \sum_{i=1}^{n} \pi\left(\sqrt{r^2-(x_i^*)^2}\right)^2 \Delta x$

taking $n \to \infty$ we get

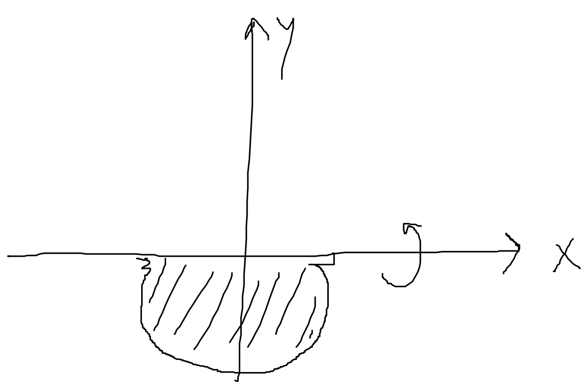
$$\int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx = \int_{-r}^{r} \pi(r^2-x^2)\,dx = \boxed{\frac{4}{3}\pi r^3}$$

Okayyy! We've learnt a new trick! If a solid can be approximated with discs, we can find its volume. What are solids that can be (easily) approximated by discs?

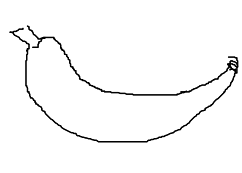
solids of revolution



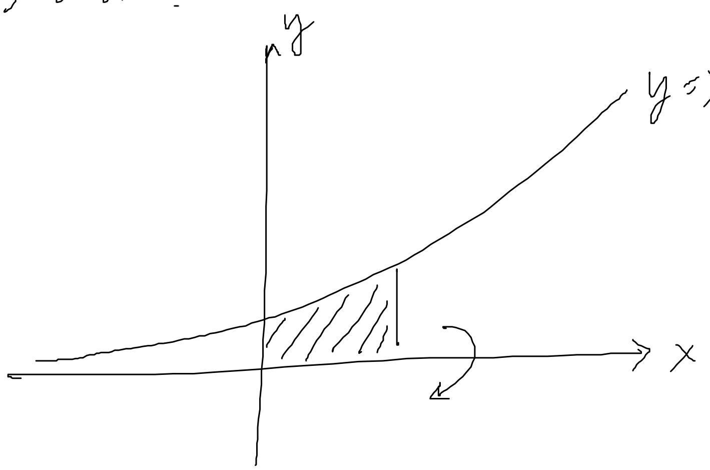
rotate around the y axis and you get a 3-d object: a pear!

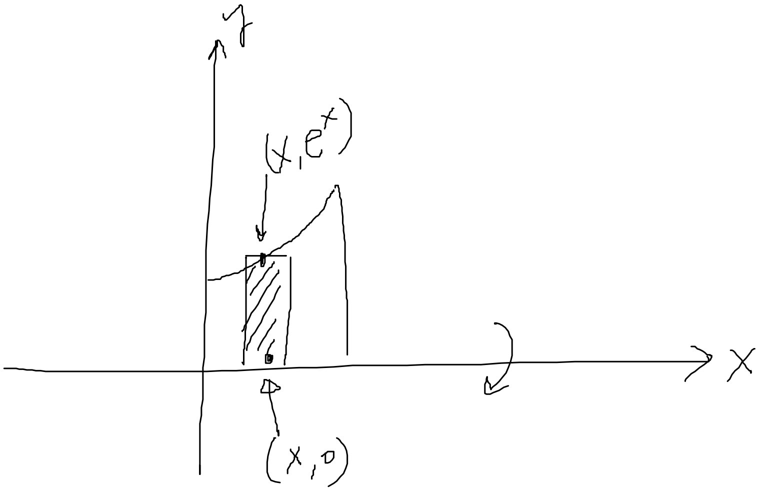


rotate around the x-axis and you get a 3-d object: an apple?

what about the banana?  poor fruit's left out!

ex #2 take the region bounded by $y=e^x$, $y=0$, $x=1$, $x=0$ and rotate it around the x-axis. What is the volume of the resulting solid?





rotate this rectangle about the x-axis and you'll get a disk. Its volume is

$$\pi(e^x)^2\Delta x$$ since its radius is e^x.

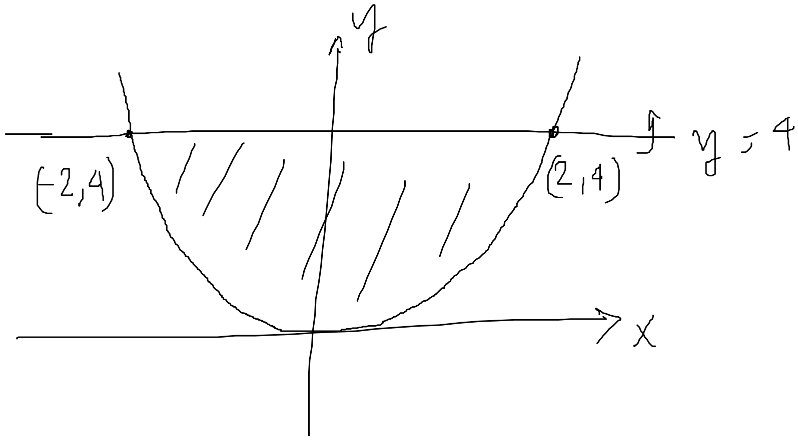
So the volume is approximated by

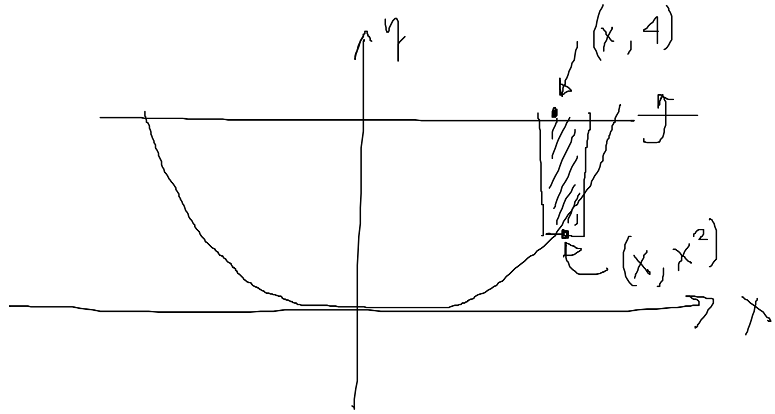
$$\sum_{i=1}^{n} \pi \left(e^{x_i^*}\right)^2 \Delta x$$

taking $n \to \infty$, the volume equals $\displaystyle\int_0^1 \pi e^{2x}\,dx =$

$$\boxed{\frac{\pi}{2}\left(e^2 - 1\right)}$$

ex #12 take the region bounded by $y = x^2$ & $y = 4$ and rotate about $y = 4$. Find its volume





rotate this rectangle about $y=4$ and you get a disc of height Δx and radius $4 - x^2$

so volume $\approx \sum_{i=1}^{n} \pi\left(4-(x_i^*)^2\right)^2 \Delta x$

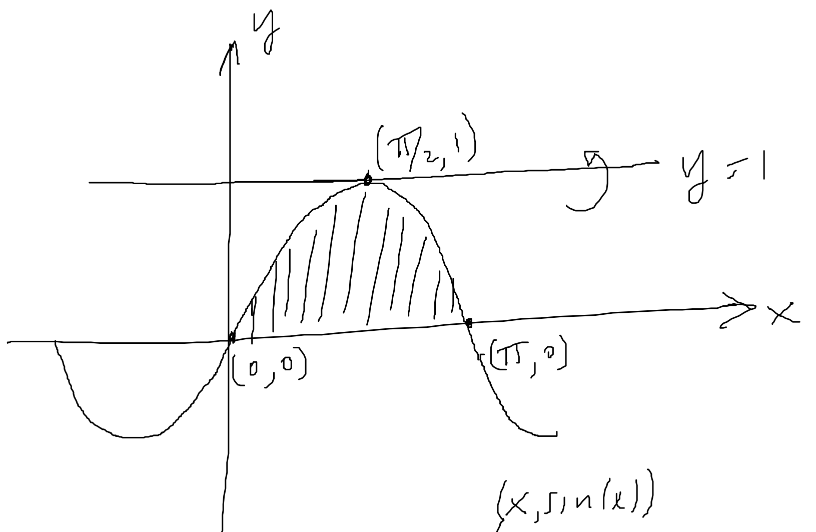
taking $n \to \infty$

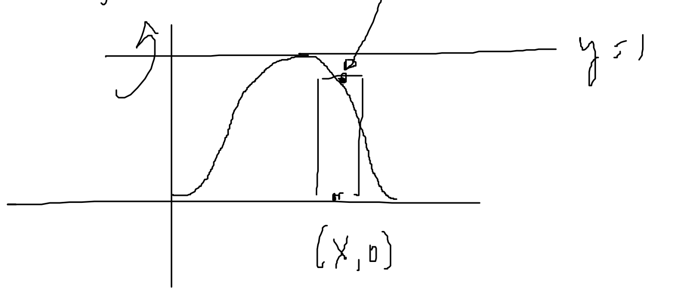
$$\int_{-2}^{2} \pi\left(4-x^2\right)^2 dx = \boxed{\frac{512}{15}\pi}$$

ex #33 rotate the region bounded by

$y=0$; $y=\sin(x)$, $x=0$, $x=\pi$

about the line $y=1$ find its volume





vol of washer =

$\pi\left(1^2 - (1-\sin(x))^2\right)$

So

$$\text{volume} = \int_0^{\pi} \pi\left(1-(1-\sin(x))^2\right)dx$$

$$= \pi\int_0^{\pi} 2\sin(x) - \sin^2(x)\,dx$$

$$= \boxed{4\pi - \pi^2/2}$$

Above, I used

$$\int \sin^2(x)\,dx = \frac{x}{2} - \frac{1}{2}\cos(x)\sin(x) + C$$

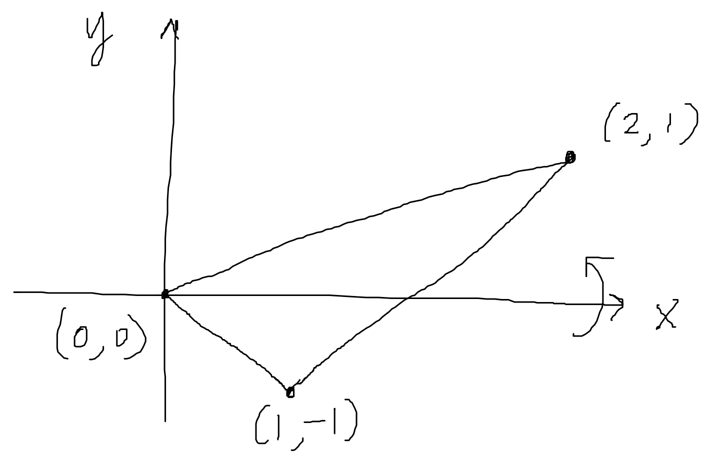
which follows from integration by parts. You'll learn it soon ☺

Here's a hard one!

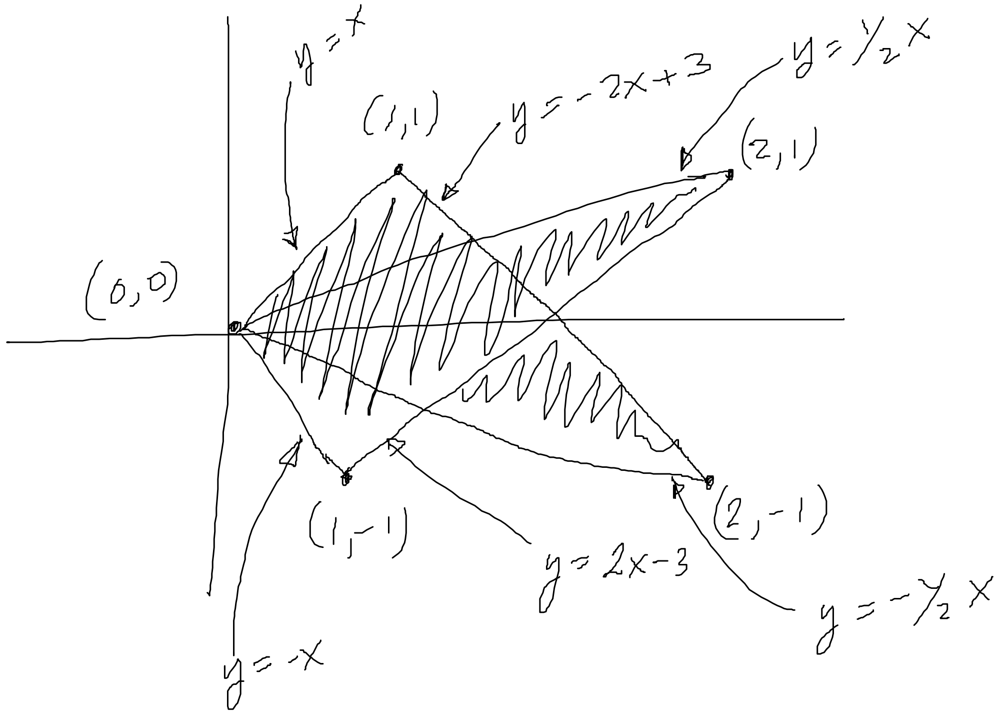
rotate the region bounded by

$$y = x/2,\quad y = -x,\quad y = 2x-3$$

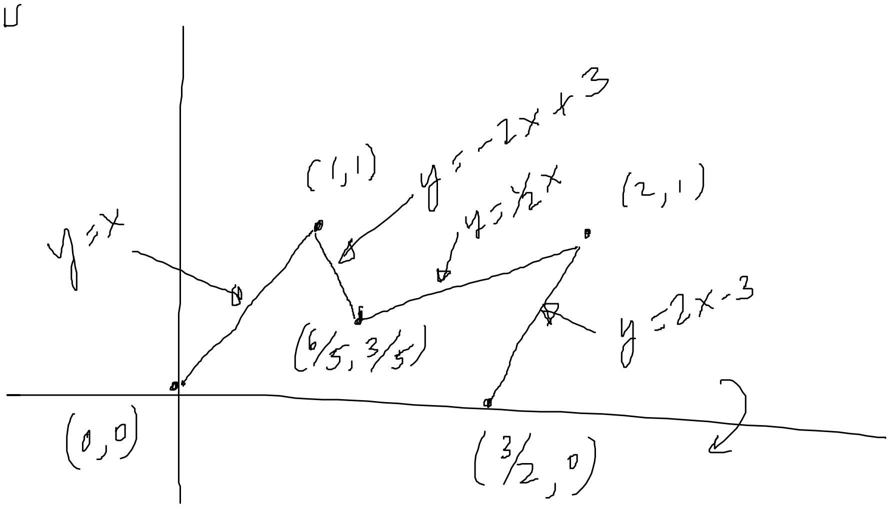
about the x-axis



rotating this, you get something whose intersection with the xy plane is



this is



we get 4 integrals!! 3 with disks, one with washers

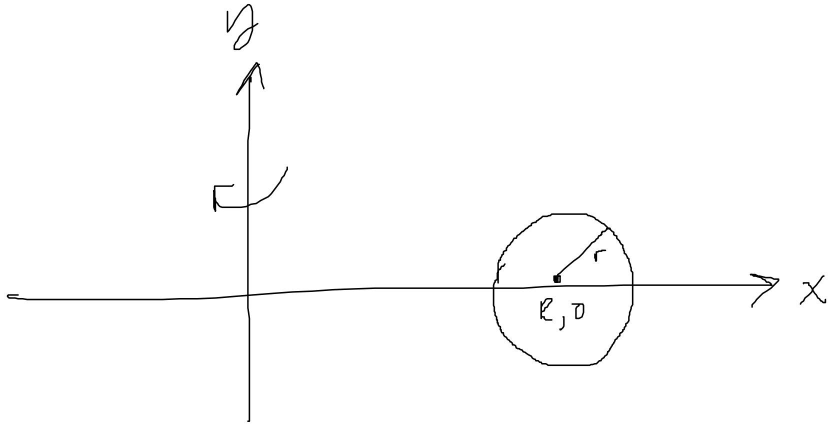
$$\int_0^1 \pi (x)^2\,dx + \int_1^{6/5} \pi(-2x+3)^2\,dx + \int_{6/5}^{3/2} \pi\left(\frac{x}{2}\right)^2 + \pi\int_{3/2}^{2} \left(\frac{x}{2}\right)^2 - (2x-3)^2\,dx = \boxed{\frac{41\pi}{50}}$$

Finally,

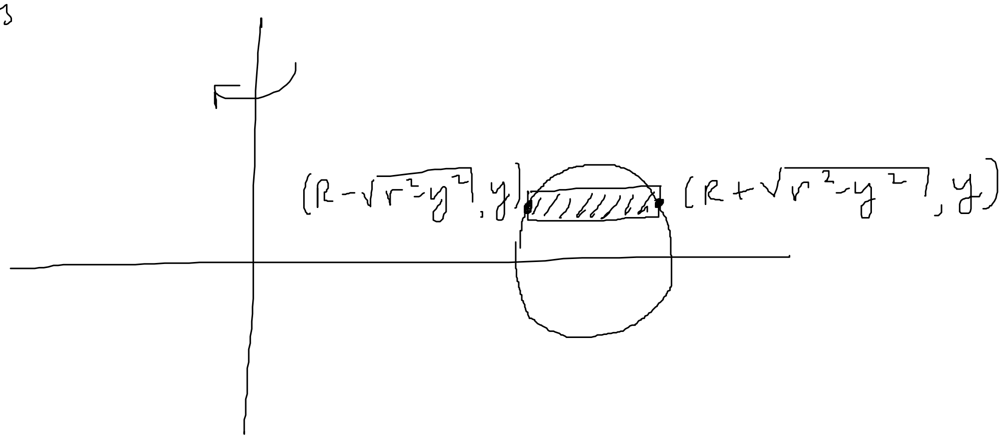
rotate the region bounded by

$$(x-R)^2 + y^2 = r^2$$

about y-axis and find its volume.



You get a torus! We'll use the method of washers



washer volume =

$$\pi\left(\left(R+\sqrt{r^2-y^2}\right)^2 - \left(R-\sqrt{r^2-y^2}\right)^2 \right) \Delta y$$

$$= \pi 4R \sqrt{r^2-y^2}\, \Delta y$$

$$\Rightarrow \text{vol} = \int_{-r}^{r} \pi 4R\sqrt{r^2-y^2}\,dy = \boxed{2\pi^2 R r^2}$$

Above, I used

$$\int \sqrt{r^2-y^2}\,dy = \frac{y}{2}\sqrt{r^2-y^2} + \frac{r}{2}\sin^{-1}\left(\frac{y}{r}\right) + C$$

this follows by the method of trig substitution, which you'll learn soon. ☺