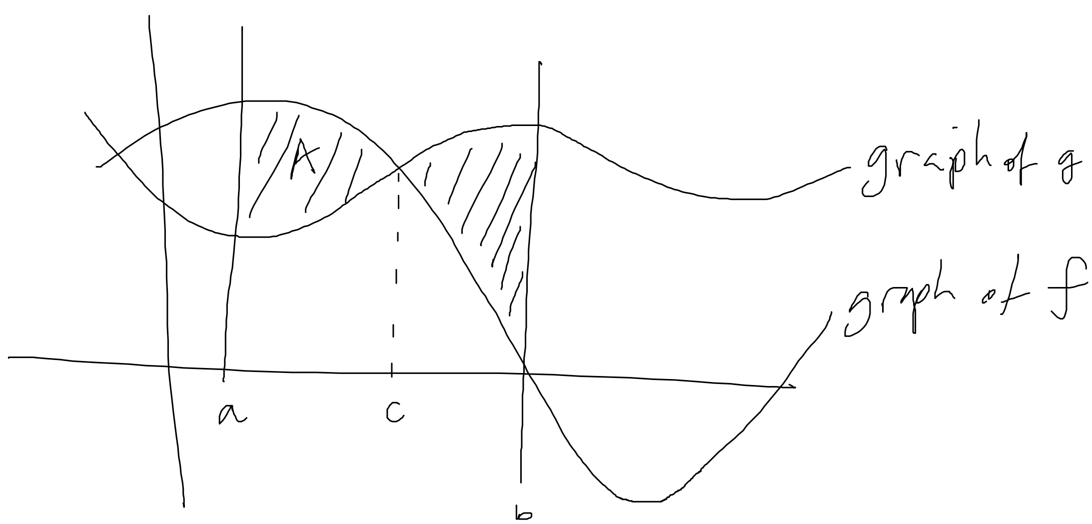
Mat 135, Jan 17 2005

§ 6.1 Areas Between Curves

we want to find A the area of the region bounded by the 4 curves

$$y = f(x), \quad y = g(x), \quad x = a, \quad x = b$$



approximating rectangles.

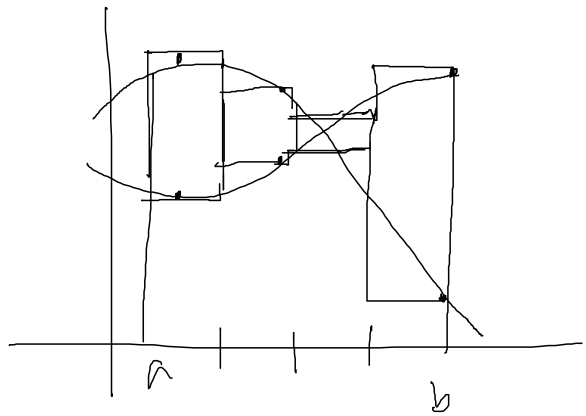
In the case $n=4$ and I chose 4 sample points x_1^* in $[x_0, x_1]$

x_2^* in $[x_1, x_2]$

x_3^* in $[x_2, x_3]$

x_4^* in $[x_3, x_4]$

$n = 4$



note: $f(x_1^*) > g(x_1^*)$ $\quad$ $f(x_3^*) < g(x_3^*)$

$f(x_2^*) > g(x_2^*)$ $\quad$ $f(x_4^*) < g(x_4^*)$

area of the first rectangle is

$(f(x_1^*) - g(x_1^*))\,\Delta x$ since height $= f(x_1^*) - g(x_1^*)$

second rectangle has area

$(f(x_2^*) - g(x_2^*))\,\Delta x$

and so on. So for this particular case, the area A is approximated by

$$(f(x_1^*) - g(x_1^*))\,\Delta x + (f(x_2^*) - g(x_2^*))\,\Delta x + (g(x_3^*) - f(x_3^*))\,\Delta x + (g(x_4^*) - f(x_4^*))\,\Delta x$$

$$= \sum_{i=1}^{4} |f(x_i^*) - g(x_i^*)|\,\Delta x$$

check out those absolute values!!

look familiar? ☺

as $n \to \infty$ the approximation will get closer & closer to A as long as

$|f-g|$ satisfies "$|f-g|$ is continuous on $[a,b]$".

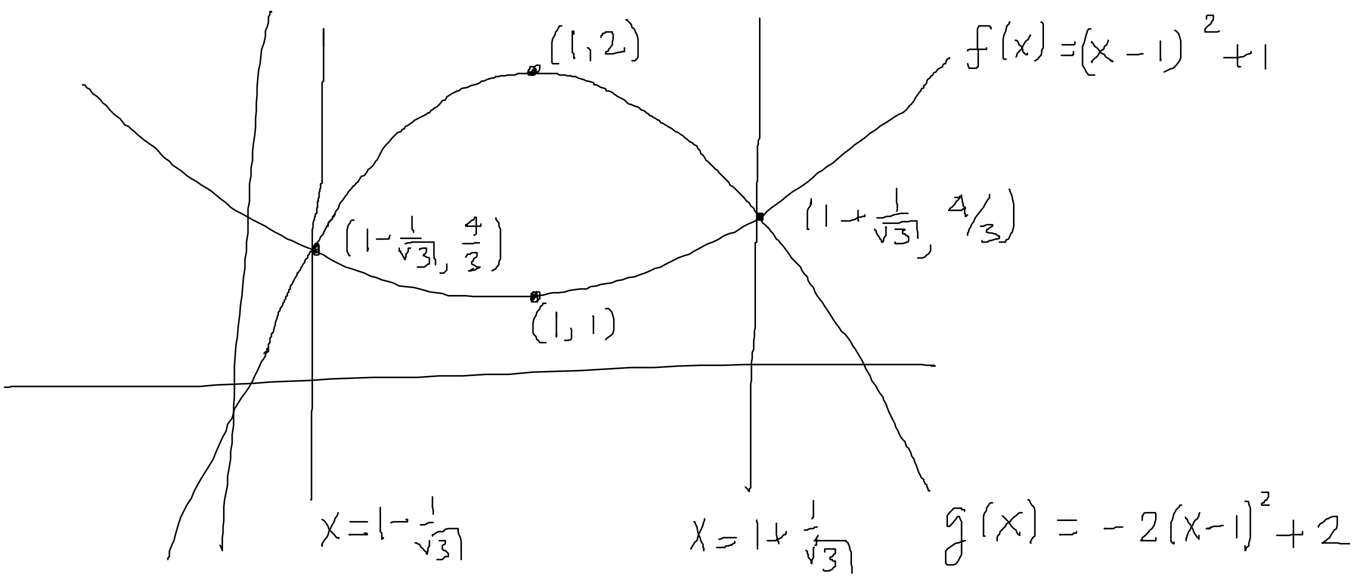
Although "$|f-g|$ is continuous except for finitely many jump or removeable discontinuities" will also do.

<u>result</u>: Assume $|f-g|$ is continuous except for finitely many jump or removeable discontinuities. Then the area between the curves $y=f(x)$, $y=g(x)$, $x=a$, and $x=b$ is

$$\int_a^b |f(x)-g(x)|\, dx.$$

<u>Note</u>: if $f \geq g$ on $[a,b]$ then the above formula equals $\int^b f(x)-g(x)\, dx$

Note, the area can also be something where the $x=a$ and $x=b$ curves aren't really relevant:



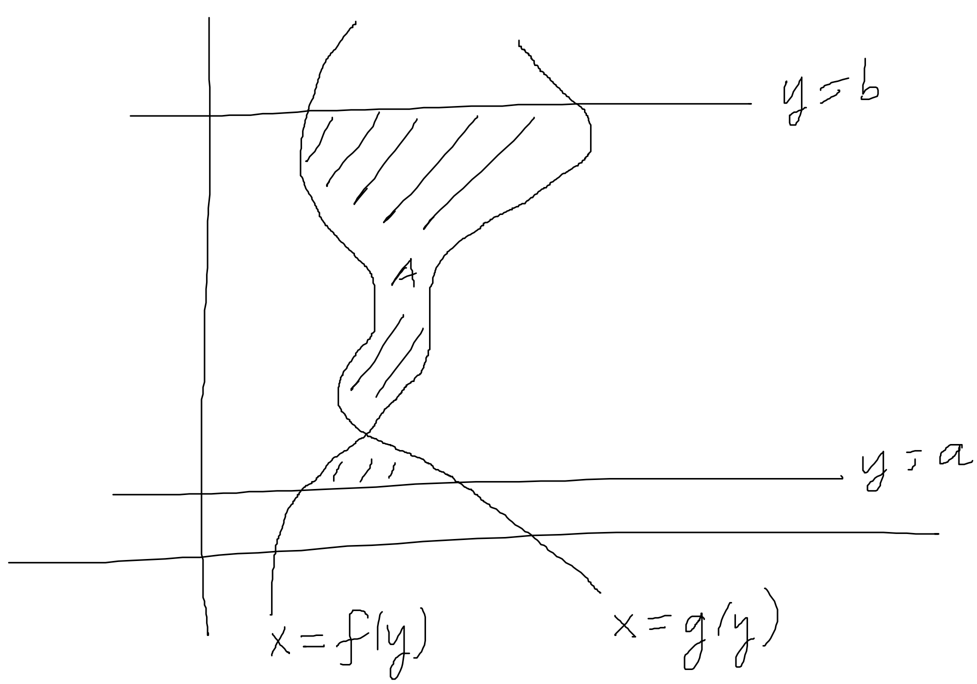
Problem "Find the region bounded by the curves $y=(x-1)^2+1$ and $y=-2(x-1)^2+2$"

<u>Ans:</u> Graph the curves and find the intersection points. The desired region is bounded by $y=f(x)=(x-1)^2+1$, $y=g(x)=-2(x-1)^2+2$, $x=a$ and $x=b$ where $a=1-1/\sqrt{3}$ and $b=1+1/\sqrt{3}$. So by our previous result

$$A = \int_{1-1/\sqrt{3}}^{1+1/\sqrt{3}} \left(-2(x-1)^2+2\right) - \left((x-1)^2+1\right)\, dx$$

$$= \boxed{\frac{4}{9}\sqrt{3}}$$

Also, regions can be bounded by functions of y just as easily as functions of x.

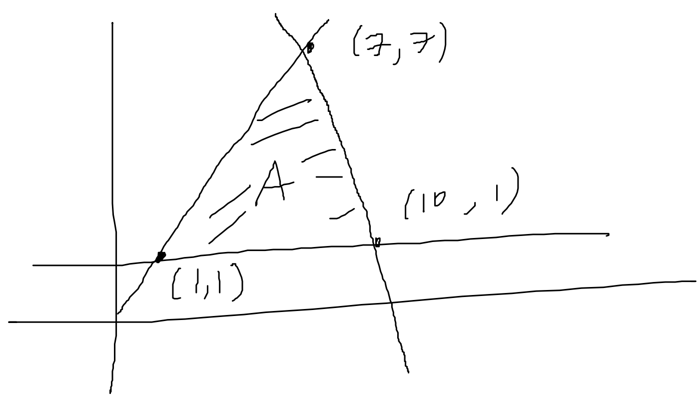


drawing horizontal rectangles of height Δy and width $|f(y_i^*) - g(y_i^*)|$ we find approximate areas and take $n \to \infty$ to find result

Assume $|f-g|$ is continuous on $[a,b]$ except for finitely many jump or removable discontinuities. Then the area between the curves $x=f(y)$, $x=g(y)$, $y=a$, and $y=b$ is

$$\int_a^b |f(y)-g(y)|\,dy.$$

ex:



Find the area bounded by the lines

$y=1$, $y=x$, and $y=-2x+21$

line 1   line 2   line 3

line 1 intersects line 2 $\Rightarrow$ $1=y=x \Rightarrow x=1 \Rightarrow y=1$ $(1,1)$

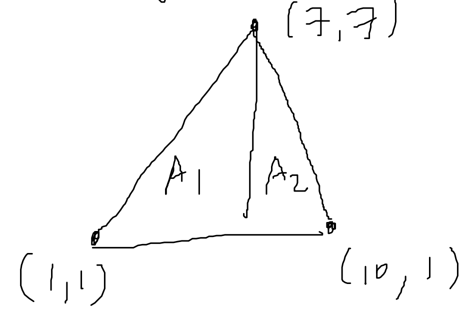
line 1 intersects line 3 $\Rightarrow$ $1=y=-2x+21 \Rightarrow x=10$ $(10,1)$

line 2 intersects line 3 $\Rightarrow$ $x=y=-2x+21 \Rightarrow x=7 \Rightarrow y=7$ $(7,7)$

geometrically, we know

$$A = \frac{1}{2}(10-1)\cdot(7-1) = 27$$

as a dx integral we have two regions to keep track of:

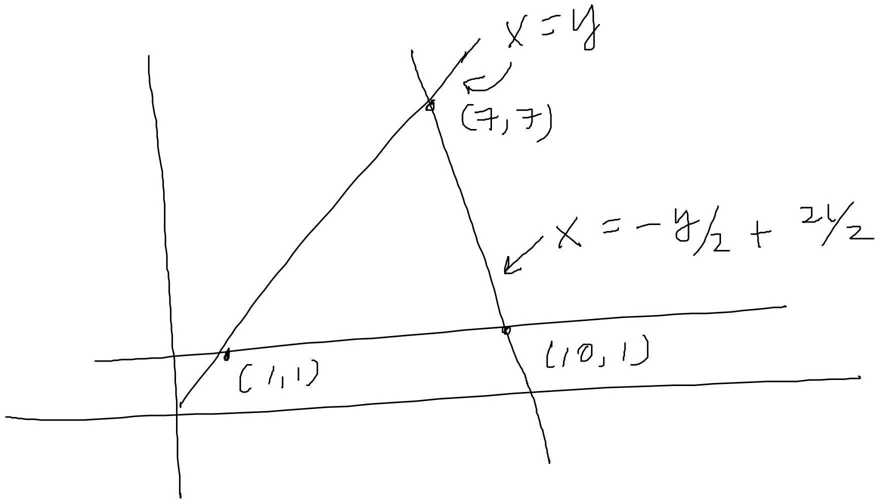


$$A_1 = \int_1^7 x-1 \, dx = 18$$

$$A_2 = \int_7^{10} (-2x+21)-1 \, dx = 9$$

$$A = A_1 + A_2 = 18 + 9 = 27$$

As a dy integral, we don't have to break the region up into two pieces. But we do need to write the bounding graphs as functions of y.

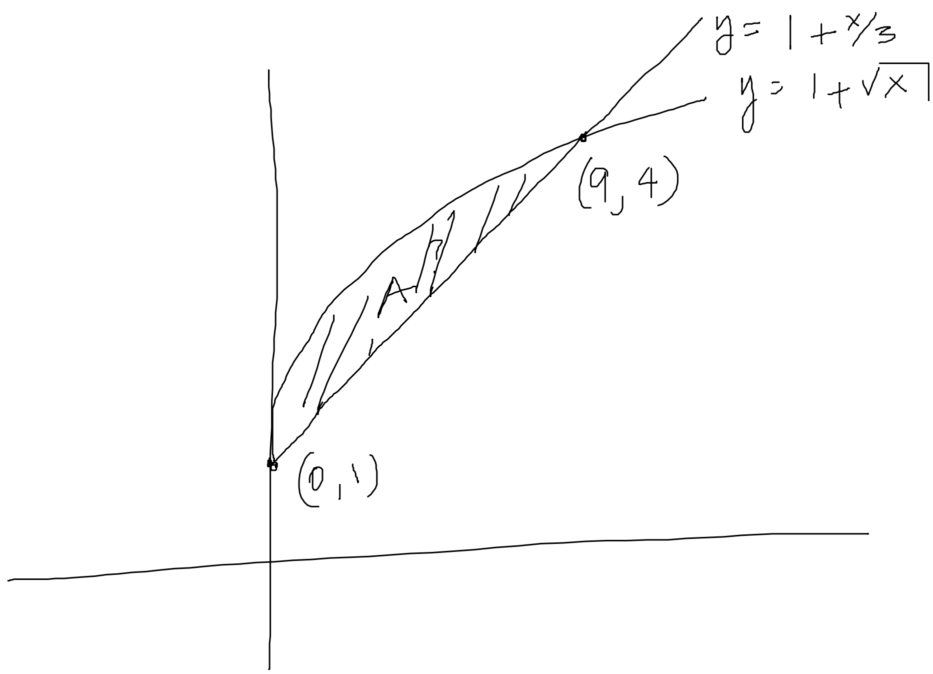


So $A = \int_1^7 \left(-y/2 + 21/2\right) - (y)\, dy = \boxed{27}$

ex: 10 sketch region enclosed by the curves

$y = 1 + \sqrt{x}$ and $y = \frac{3+x}{3} = 1 + \frac{1}{3}x$

and find its area



to find A, we need to find the points where the curves cross. That is, find x where

$$1 + \frac{x}{3} = 1 + \sqrt{x} \Rightarrow \frac{x}{3} = \sqrt{x} \Rightarrow \frac{x^2}{9} = x$$

$$\Rightarrow x^2 - 9x = 0 \Rightarrow x(x-9) = 0$$

$$x = 0 \quad \text{and} \quad x = 9$$

$$\text{Area} = \int_0^9 1 + \sqrt{x} - (1 + x/3)\,dx = \int_0^9 \sqrt{x} - \frac{x}{3}\,dx$$

$$= \frac{2}{3}x^{3/2} - \frac{x^2}{6}\Big|_0^9 = \boxed{9/2}$$

ex 17: same question

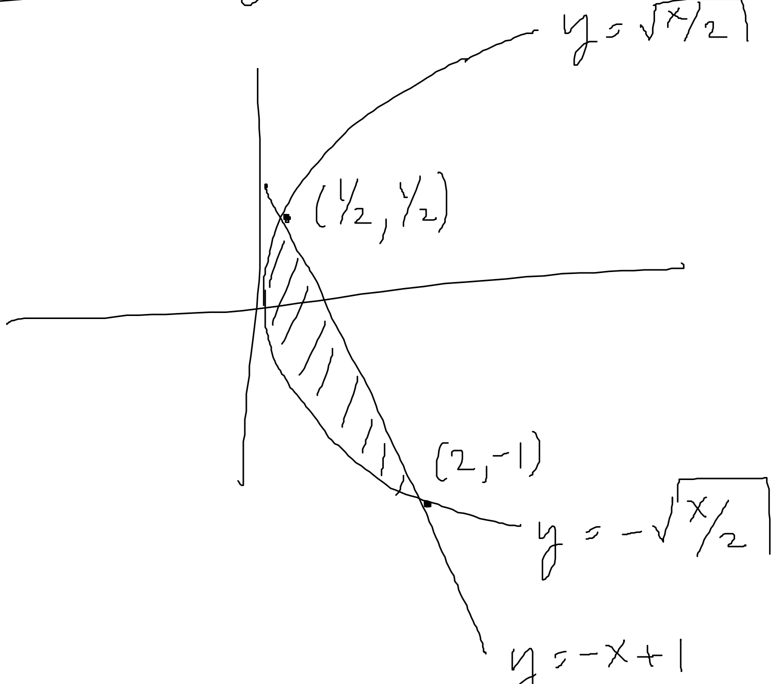
$$x = 2y^2 \qquad x + y = 1$$

here it's a choice.

method 1: find things in terms of functions of x

method 2: find things in terms of functions of y.

method 1: $y = \pm\sqrt{\frac{x}{2}}$ $\quad y = -x + 1$



top intersection point $\sqrt{x/2} = -x + 1 \Rightarrow x = 1/2$

bottom intersection point $-\sqrt{x/2} = -x + 1 \Rightarrow x = 2$

then the desired area is

$$\int_0^{1/2} \sqrt{x/2} - \left(-\sqrt{x/2}\right) dx + \int_{1/2}^{2} (-x+1) - \left(-\sqrt{x/2}\right) dx$$

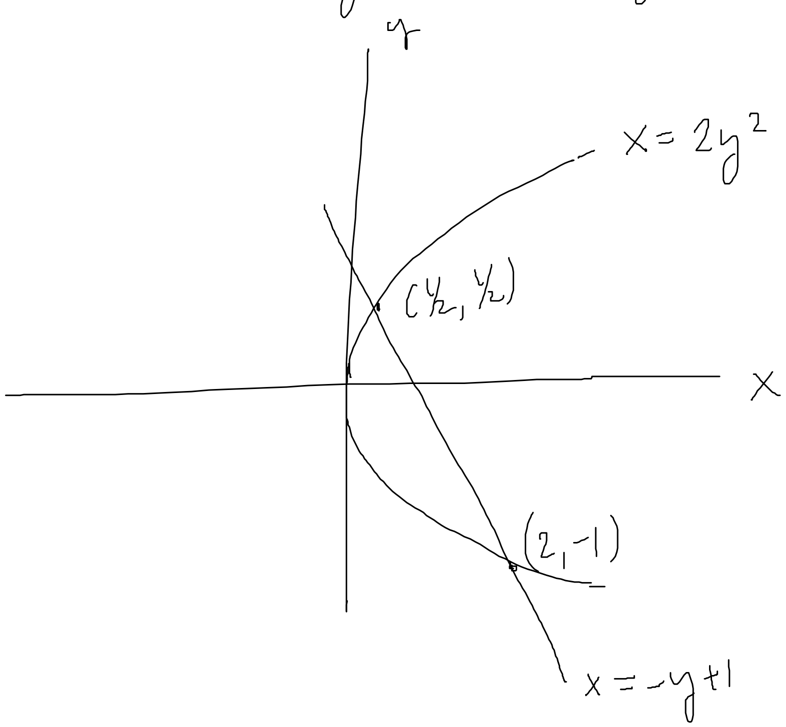
$$= \int_0^{1/2} 2\sqrt{x/2} + \int_{1/2}^{2} -x+1+\sqrt{x/2}\, dx =$$

$$\frac{2}{3}\sqrt{2}\, x^{3/2}\Big|_0^{1/2} + \left(-\frac{1}{2}x^2 + x + \frac{1}{3}\sqrt{2}\, x^{3/2}\right)\Big|_1^2$$

$$= \boxed{9/8}$$

Method 2: write both functions in terms of y

$x = 2y^2 \qquad x = -y+1$



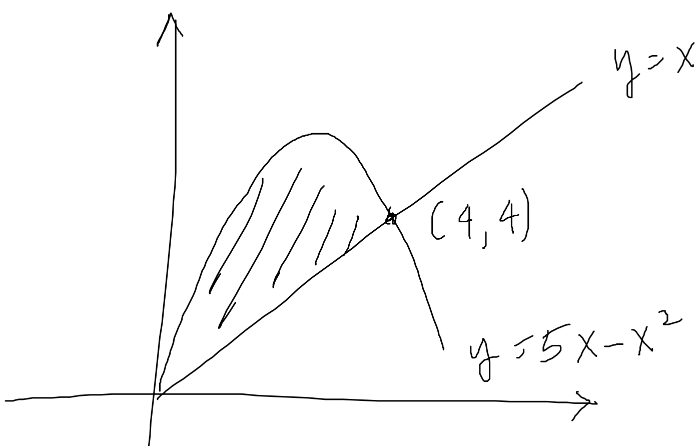
And so from before,

$$\text{area} = \int_{-1}^{1/2} (-y+1) - (2y^2)\,dy$$

$$= -\frac{1}{2}y^2 + y - \frac{2}{3}y^3 \Big|_{-1}^{1/2} = \boxed{9/8} \quad ✓$$

we got the same answer!

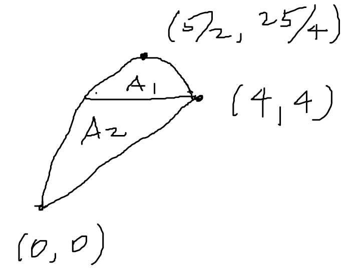
ex 1:



better as dx integral

$$\text{Area} = \int_0^4 (5x - x^2) - (x)\,dx = \boxed{\frac{32}{3}}$$

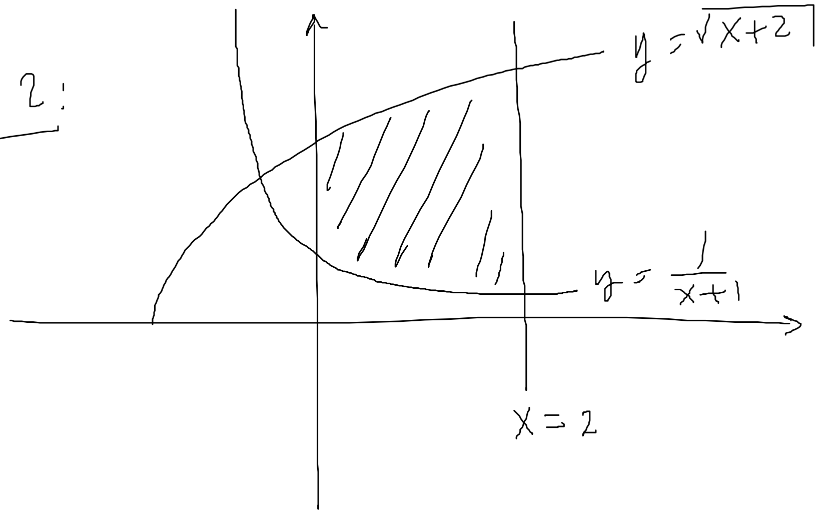
As a dy integral:



$$A_1 = \int_4^{25/4} \left(\tfrac{5}{2} + \sqrt{25-4y}\right) - \left(\tfrac{5}{2} - \sqrt{25-4y}\right) dy = 9/2$$

$$A_2 = \int_0^4 (y) - \left(\tfrac{5}{2} - \sqrt{25-4y}\right) dy = 37/6 \qquad A_1 + A_2 = \frac{32}{3}$$

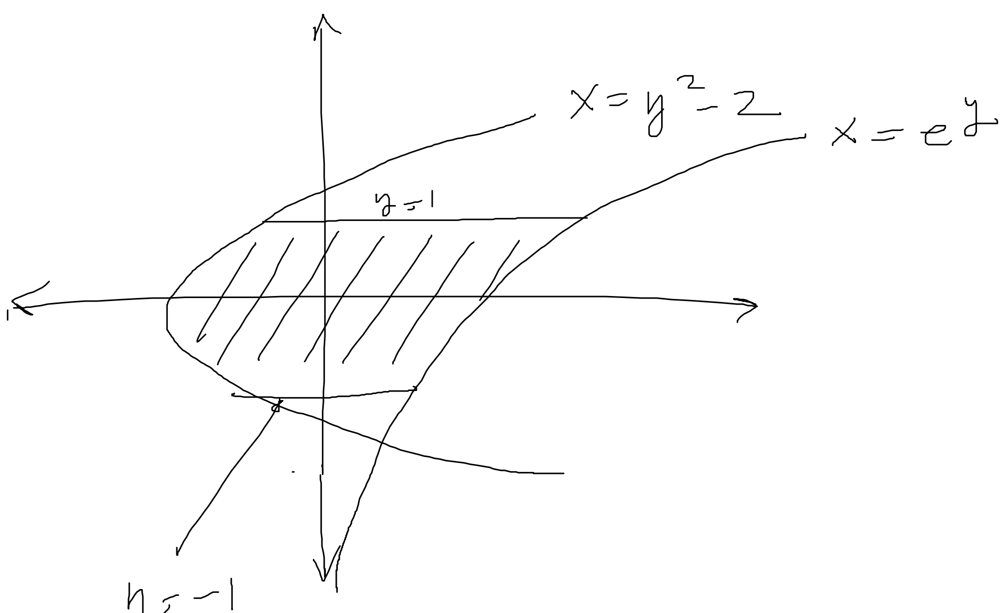
ex 2:



better as dx integral

$$\int_0^2 \sqrt{x+2} - \frac{1}{x+1}\,dx = \boxed{\frac{16}{3} - \frac{4\sqrt{2}}{3} - \ln(3)}$$

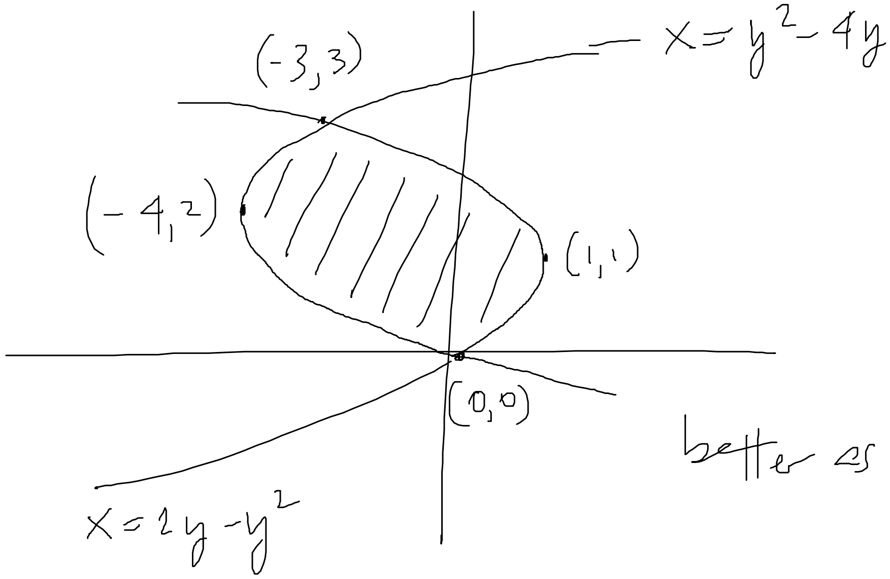
ex 3



better as a dy integral

$$A = \int_{-1}^{1} e^y - (y^2 - 2)\,dy = \boxed{\frac{10}{3} + e - \frac{1}{e}}$$

ex 4

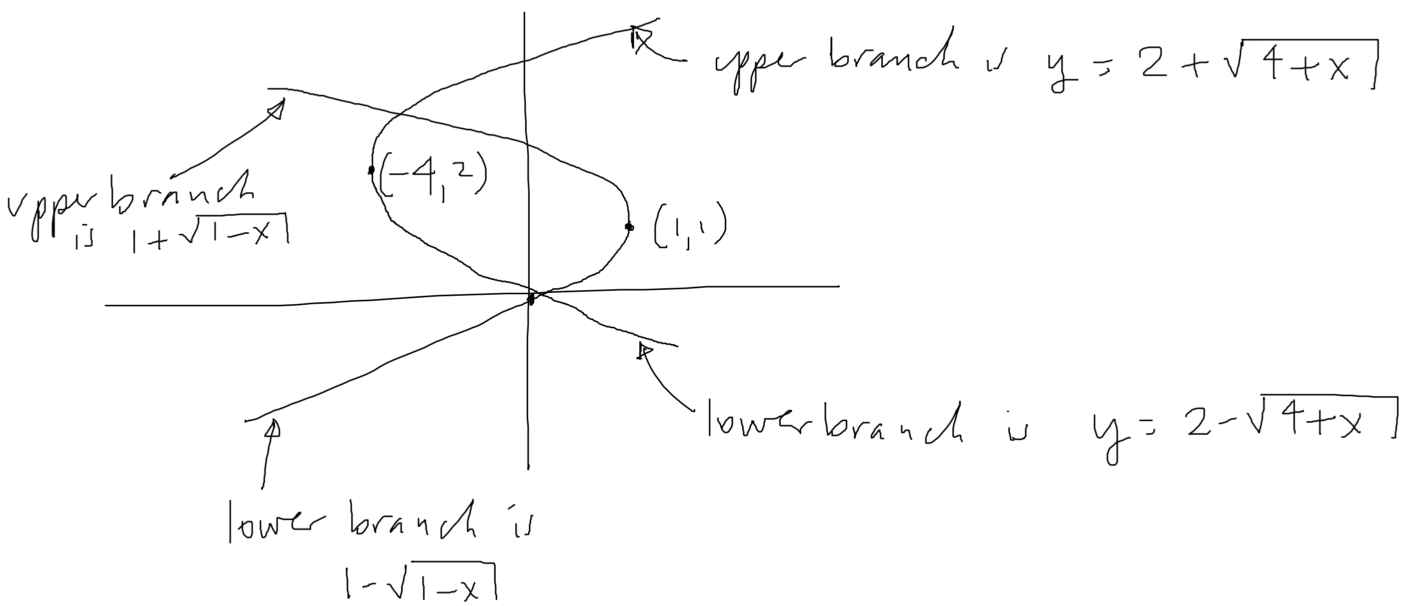


better as dy integral

$$A = \int_0^3 (2y - y^2) - (y^2 - 4y)\, dy = \boxed{9}$$

What if you'd insisted on doing that as a dx integral?

First write the curves as functions of x.



we get 3 integrals $x = -4$ to $x = -3$

$x = -3$ to $x = 0$

$x = 0$ to $x = 1$

$$A = \int_{-4}^{-3} (2+\sqrt{4+x}) - (2-\sqrt{4+x})\,dx + \int_{-3}^{0} (1+\sqrt{1-x}) - (2-\sqrt{4+x})\,dx$$

$$+ \int_{0}^{1} (1+\sqrt{1-x}) - (1-\sqrt{1-x})\,dx = \boxed{9}$$