Mat 135 Jan 14 2005

## §5.6 The Logarithm defined as an Integral

Recall that in Chapter 1, we defined $e^x$ and studied it and then defined $\ln(x)$ as the inverse of $\exp(x)$.

Step 1: define & understand $\exp(x)$

Step 2: define "$\ln(x)$ is the inverse of $\exp(x)$"

Step 3: understand $\ln(x)$.

Now, we'll do something different. We'll define $\ln(x)$ on its own merits and will do the following

Step 1: define $\ln(x)$ & understand its properties

Step 2: define exp via "exp is the inverse of ln"

Step 3: understand exp.

Q: Why bother ?!? We already did all that stuff in Chapter 1! Who needs to know that there are two completely different ways of creating the exact same thing?

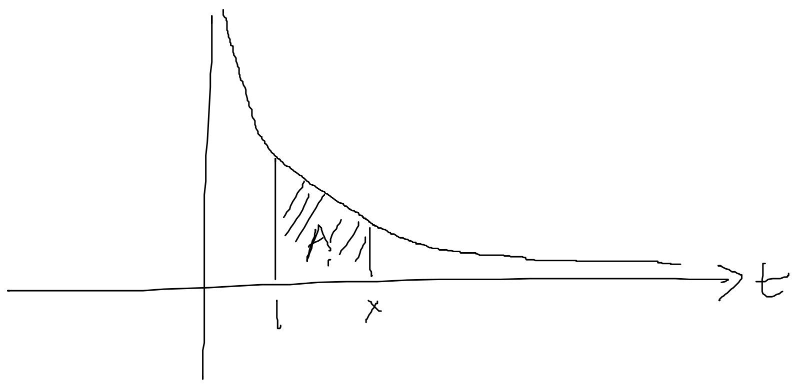
A: 1) There was a bunch of vague & heuristic stuff in chapter 1 that we glossed over for example in just because you understand how to define $2^x$ when $x$ is an integer or a rational number, how do you really understand what $2^{\sqrt{2}}$ mean?

2) The alternate definition of ln will show us some interesting things about when functions can be integrated or not.

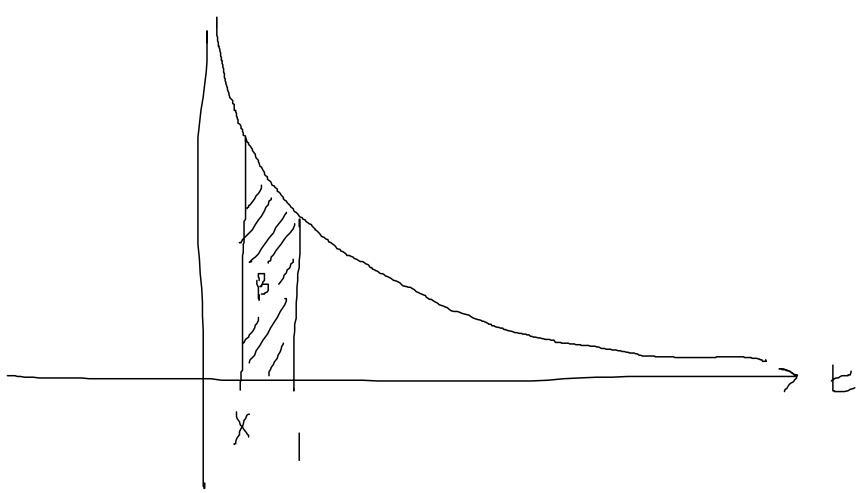
<u>definition</u>: given $x>0$, we define

$$\ln(x) = \int_1^x \frac{1}{t}\,dt$$

That is, if $x>1$ then $\ln(x)$ is the area A




if $x<1$ then $\ln(x)$ is the negative of the area B



You see immediately from the definition the following:

$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$$

$$\ln(x) > 0 \quad \text{if } x > 1$$

$$\ln(x) < 0 \quad \text{if } x < 1$$

$$\frac{d}{dx}\ln(x) = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x} \quad \text{by F.T. of C.}$$

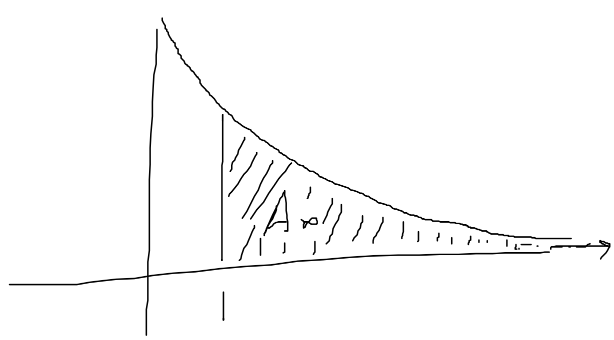
Now for two interesting questions.

From chapter 1, you know that

$$\lim_{x\to\infty} \ln(x) = \infty$$

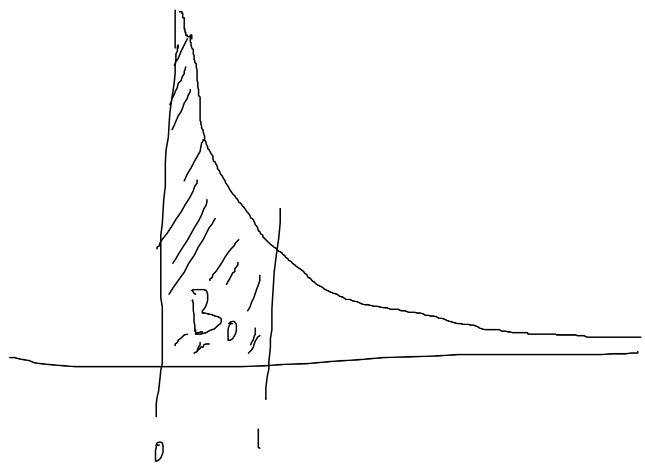
$$\lim_{x\to 0^+} \ln(x) = -\infty$$

That means that $A_\infty$ the area under $f(t) = \frac{1}{t}$ from $t=1$ to $t=\infty$ must be infinite



So even though $f(t) = \frac{1}{t} \to 0$ as $t \to \infty$ and the graph of $f$ is getting closer & closer to the $t$-axis as $t \to \infty$, there's an infinite amound of area under the graph.

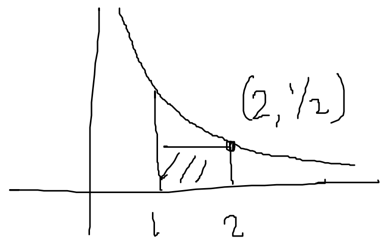
Similarly, $B_0$ the area under the graph of $f(t) = \frac{1}{t}$ from $t = 0$ to $t = 1$ must also be infinite




So even though there's a vertical asymptote at $t = 0$ and the graph of $f(t) = \frac{1}{t}$ is getting closer & closer to the $y$-axis, there's an infinite amount of area in there.

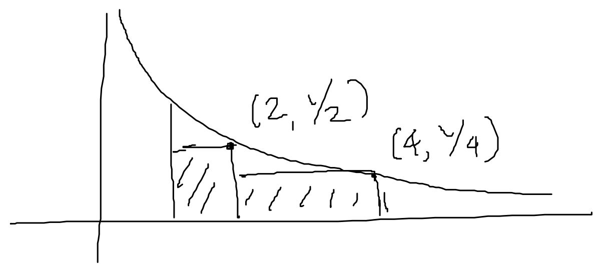
How can we understand these areas being infinite?

$$A_2 = \int_1^2 \frac{1}{t}\,dt$$



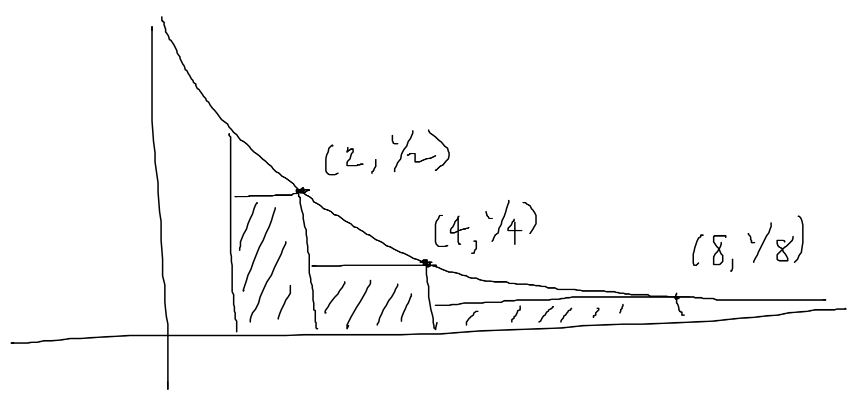

$$A_2 > (2-1)\frac{1}{2} = 1/2$$

$$A_4 = \int_1^4 \frac{1}{t}\,dt$$




$$A_4 > (2-1)\frac{1}{2} + (4-2)\frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

$$A_8 = \int_1^8 \frac{1}{t}\,dt$$




$$A_8 > (2-1)\frac{1}{2} + (4-2)\frac{1}{4} + (8-4)\frac{1}{8}$$

$$= 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} = 3/2$$

In general,

$$A_{2^n} > (2-1)\frac{1}{2} + (4-2)\frac{1}{4} + \dots + (2^n - 2^{n-1})\frac{1}{2^n} = \frac{n}{2}$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{n \text{ terms in the sum}}$$

Note: $A_{2^n} = \ln(2^n)$

We've just proven that

$$\lim_{n\to\infty} \ln(2^n) = \lim_{n\to\infty} A_{2^n} > \lim_{n\to\infty} \frac{n}{2} = \infty$$

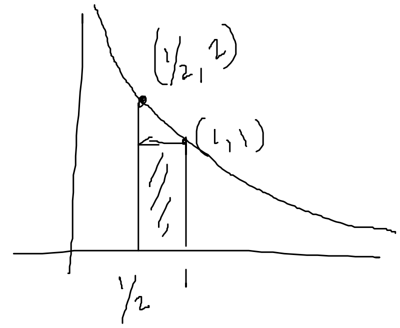
So there must be an infinite amount of area under the graph of $\frac{1}{t}$ from $t=1$ to $t=\infty$ because the area from $t=1$ to $t=2^n$ is getting bigger & bigger.

---

What about $B_0$ the area from $t=0$ to $t=1$?

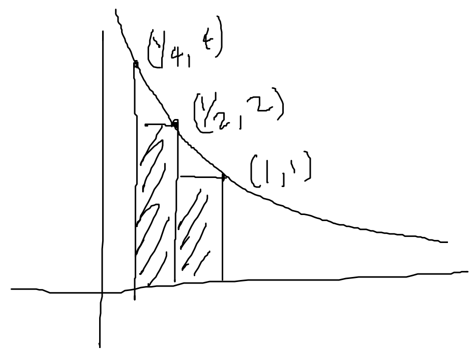
Same idea

$$B_{\frac{1}{2}} > \left(1-\frac{1}{2}\right)\cdot 1 = \frac{1}{2}$$




$$B_{1/4} > \left(1-\frac{1}{2}\right)\cdot 1 + \left(\frac{1}{2}-\frac{1}{4}\right)\cdot 2$$

$$= \frac{1}{2} + \frac{1}{4}\cdot 2 = 1$$




$$B_{\frac{1}{2^n}} > \underbrace{\left(1-\frac{1}{2}\right)\cdot 1 + \left(\frac{1}{2}-\frac{1}{4}\right)\cdot 2 + \dots + \left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)2^{n-1}}_{n \text{ terms}} = \frac{n}{2}$$

So as before we see that

$$\lim_{n\to\infty} -\ln\left(\frac{1}{2^n}\right) = \lim_{n\to\infty} B_{\frac{1}{2^n}} = \infty$$ so there's

an infinite amount of area under $f(t) = \frac{1}{t}$ from $t=0$ to $t=1$.

Using $\frac{d}{dx}\ln(x) = \frac{1}{x}$ (from F.T. of C.)

we can prove the usual rules:

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\ln(x^r) = r\ln(x)$$

ex: let $f(x) = \ln(xy)$

then $\frac{df}{dx} = \frac{d}{dx}\ln(xy) = \frac{1}{xy}\cdot y$ by chain rule

$$= \frac{1}{x}$$

$\Rightarrow \ln(xy)$ is an antiderivative of $\frac{1}{x}$.

$\Rightarrow \ln(xy) = \ln(x) + C_1$ for some $C_1$

How do we find $c$?

try evaluating at a particular value of $x$. we know $\ln(1) = 0$ so evaluating at $x=1$ will give a nice simplification

$$\ln(1\cdot y) = \ln(1) + c = c \quad \text{since } \ln(1) = 0$$

$$\Rightarrow \ln(y) = c \; !$$

we've just shown that

$$\ln(xy) = \ln(x) + \ln(y).$$

See book for $\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$.

---

for $\ln(x^r) = r\ln(x)$, let

$f(x) = \ln(x^r) - r\ln(x)$. Then $\frac{df}{dx} = \frac{1}{x^r}(rx^{r-1}) - r\frac{1}{x} = 0$

$\Rightarrow f(x) = c$ some $c$. evaluate at $x=1$

$$f(1) = c = \ln(1^r) - r\ln(1) = 0 - 0 = 0 \Rightarrow c = 0$$

$$\Rightarrow \ln(x^r) - r\ln(x) = 0 \Rightarrow \ln(x^r) = r\ln(x)$$