Mat 135 Jan 12, 2005

In yesterday's lecture, I did a number of examples like

$$\int 2\sin(x)\cos(\cos(x)+\pi) = -2\sin(\cos(x)+\pi) + C$$

I did this by viewing the integrand as something that might come from the chain rule.

Remember the C.R.:
confronted with

$\frac{d}{dx}\sin(\cos(x)+\pi)$ you viewed this as

" $\frac{d}{dx}\sin(u(x))$ where $u(x) = \cos(x)+\pi$ "

and then by the chain rule,

$$\frac{d}{dx}\sin(u(x)) = \frac{d}{du}\sin(u)\,\frac{du}{dx} \quad \text{where } u(x) = \cos(x)+\pi$$

$$= \cos(u)\,[-\sin(x)] \quad \text{where } u(x) = \cos(x)+\pi$$

$$= -\sin(x)\cos(\cos(x)+\pi)$$

and so

$$\frac{d}{dx}\sin(\cos(x)+\pi) = -\sin(x)\cos(\cos(x)+\pi).$$

At this point, this (hopefully) feels like second nature. But remember... it took experience and practice to make the conceptual leap of

"that's just $\frac{d}{dx}\sin(u(x))$ where $u(x) = \cos(x)+\pi$",

The experience & practice was: how to choose u.

The substitution rule for integration is the chain rule in disguise

fact 1: $\displaystyle\int F'(g(x))g'(x)\,dx = \int \frac{d}{dx}F(g(x))\,dx$

$$= F(g(x)) + C$$

fact 2: $\displaystyle\int F'(u)\,du = F(u) + C$

Since

$$F(u) + C = F(g(x)) + C \quad \text{if } u = g(x)$$

we see that

$$\boxed{\int f(g(x))\, g'(x)\, dx = \int f(u)\, du \quad \text{if } u = g(x) \text{ and } g \text{ is differentiable}}$$

that's the substitution rule.

ex: $\int 2\sin(x)\cos(\cos(x) + \pi)\, dx$

take $u = \cos(x) + \pi = g(x)$

then $g'(x) = -\sin(x)$ and

$$\int 2\sin(x)\cos(\cos(x) + \pi)\, dx$$

$$= -2\int \cos(g(x))\, g'(x)\, dx$$

$$= -2\int \cos(u)\, du \quad \text{where } u = \cos(x) + \pi$$

$$= -2\sin(u) \quad \text{where } u = \cos(x) + \pi$$

$$= -2\sin(\cos(x)+\pi) + C_1$$

this shows that as before,

$$\int 2\sin(x)\cos(\cos(x)+\pi)\,dx = -2\sin(\cos(x)+\pi) + C_1.$$

ex: $\displaystyle\int \frac{\ln(x)}{x}\,dx$

try $u = \ln(x) = g(x)$

then $g'(x) = \frac{1}{x}$ and

$$\int \frac{\ln(x)}{x}\,dx = \int g(x)\,g'(x)\,dx$$

$$= \int u\,du \quad \text{where } u = \ln(x)$$

$$= \frac{u^2}{2} + C_1 \quad \text{where } u = \ln(x)$$

$$= \frac{(\ln(x))^2}{2} + C_1$$

ex: $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$

try $u = \dfrac{\pi}{x} = g(x)$

then $g'(x) = -\dfrac{\pi}{x^2} \Rightarrow \dfrac{1}{x^2} = -\dfrac{1}{\pi} g'(x)$

$$\int \frac{\cos(\pi/x)}{x^2}\,dx = -\frac{1}{\pi}\int \cos(g(x))\,g'(x)\,dx$$

$$= -\frac{1}{\pi}\int \cos(u)\,du \quad \text{where } u = \frac{\pi}{x}$$

$$= -\frac{1}{\pi}\sin(u) + C_1 \quad \text{where } u = \frac{\pi}{x}$$

$$= \boxed{-\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + C_1}$$

Note: there's a shorthand for the above.

$$u = \frac{\pi}{x} \Rightarrow du = -\frac{\pi}{x^2}\,dx \Rightarrow -\frac{1}{\pi}\,du = \frac{1}{x^2}dx$$

$$\Rightarrow \int \cos\left(\frac{\pi}{x}\right) \underbrace{\frac{1}{x^2}dx} = -\frac{1}{\pi}\int \cos(u)\,du = -\frac{1}{\pi}\sin(u) + C_1$$

$$= -\frac{1}{\pi}\sin\left(\frac{\pi}{x}\right) + C_1$$

ex! $\int \sqrt[3]{x^3+1}\, x^5\, dx$

okay... let's guess

$$u = x^3+1 \Rightarrow du = 3x^2 dx$$

$$\Rightarrow \frac{1}{3} du = x^2 dx$$

so $\int \sqrt[3]{x^3+1}\, x^5 dx = \int \underbrace{\sqrt[3]{x^3+1}}_{u^{1/3}}\ \underbrace{x^3}_{\text{what to do ?!?}}\ \underbrace{x^2 dx}_{\frac{1}{3} du}$

well $u = x^3 + 1 \Rightarrow x^3 = u - 1$

and so

$$\int \sqrt[3]{x^3+1}\, x^5 dx = \int u^{1/3} (u-1) \frac{1}{3} du \quad \text{where } u = x^3+1$$

$$= \frac{1}{3} \int u^{4/3} - u^{1/3} du \quad \text{"} \quad \text{"}$$

$$= \frac{1}{3} \left[\frac{3}{7} u^{7/3} - \frac{3}{4} u^{4/3} + C \right] \quad \text{"} \quad \text{"}$$

$$= \boxed{\frac{1}{7}(x^3+1)^{7/3} - \frac{1}{4}(x^3+1)^{4/3} + C}$$

ex: $\displaystyle\int \frac{3x-1}{(3x^2-2x+1)^4}\,dx$

let's guess $u = 3x^2-2x+1$

$\Rightarrow du = 6x-2\,dx$

$\Rightarrow \frac{1}{2}\,du = 3x-1\,dx$

$$\Rightarrow \int \frac{3x-1}{(3x^2-2x+1)^4}\,dx = \int \frac{1}{u^4}\,\frac{1}{2}\,du \quad \text{where } u = \dots$$

$$= \frac{1}{2}\int u^{-4}\,du \quad \text{where } u = \dots$$

$$= -\frac{1}{6}u^{-3} + C \quad \text{where } u = \dots$$

$$= \boxed{-\frac{1}{6}(3x^2-2x+1)^{-3} + C}$$

ex: $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

let's guess $u = 1+x^2$

$\Rightarrow du = 2x\,dx$

$\Rightarrow \frac{1}{2}\,du = x\,dx$

So $$\int \frac{1+x}{1+x^2}\,dx = \int \underbrace{\frac{1}{1+x^2}}_{} + \underbrace{\frac{x}{1+x^2}}_{}\,dx$$

this is $\frac{d}{dx}\tan^{-1}(x)$

do this via substitution

$$= \tan^{-1}(x) + \int \frac{x}{1+x^2}\,dx$$

$$= \tan^{-1}(x) + \int \frac{1}{u}\,\frac{1}{2}\,du \quad \text{where } u = 1+x^2$$

$$= \tan^{-1}(x) + \frac{1}{2}\ln(u) + C \quad \text{" "}$$

$$= \boxed{\tan^{-1}(x) + \frac{1}{2}\ln(1+x^2) + C}$$

What about definite integrals?

$$\int_0^{\pi} x\cos(x^2)\,dx = ?$$

There are two ways to do this. You should choose whichever one you like best.

<u>First Way</u>:

$$\int_0^{\pi} x\cos(x^2)\,dx = F(x)\Big|_0^{\pi}$$ where F is an antiderivative of $x\cos(x^2)$. That is,

$$F(x) = \int x\cos(x^2)\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \frac{1}{2}\int \cos(u)\,du \quad \text{where } u = x^2$$

$$= \frac{1}{2}\sin(u) \quad \text{where } u = x^2$$

$$= \frac{1}{2}\sin(x^2)$$

Ah-hah! $F(x) = \frac{1}{2}\sin(x^2)$ and so

$$\int_0^{\pi} x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big|_0^{\pi} = \frac{1}{2}\sin(\pi^2) - \frac{1}{2}\sin(0)$$

$$= \boxed{\frac{1}{2}\sin(\pi^2)}$$

Second way.

$$\int_0^{\pi} x\cos(x^2)\,dx = ?$$

I want $u = x^2$. If the dx integral goes from $x = 0$ to $x = \pi$ then the du integral will have to go from $u = 0$ to $u = \pi^2$ and so...

$$\int_0^{\pi} x\cos(x^2)\,dx = \int_0^{\pi^2} \cos(u)\,\tfrac{1}{2}du = \tfrac{1}{2}\sin(u)\Big|_0^{\pi^2}$$

$$= \tfrac{1}{2}\sin(\pi^2) - \tfrac{1}{2}\sin(0)$$

$$= \boxed{\tfrac{1}{2}\sin(\pi^2)}$$

ex: $$\int_{1/6}^{1/2} \csc(\pi t)\cot(\pi t)\,dt$$

First way:

$$\int_{1/6}^{1/2} \csc(\pi t)\cot(\pi t)\,dt = F(t)\Big|_{1/6}^{1/2} \quad \text{where}$$

$$F'(t) = \csc(\pi t)\cot(\pi t)$$

$$F(t) = \int \csc(\pi t)\cot(\pi t)\,dt = \int \csc(u)\cot(u)\,\frac{1}{\pi}\,du$$

where $u = \pi t$

$$= -\frac{\csc(u)}{\pi} + C \quad \text{where } u = \pi t$$

$$= -\frac{\csc(\pi t)}{\pi} + C$$

and so $\displaystyle\int_{1/6}^{1/2} \csc(\pi t)\cot(\pi t)\,dt = -\frac{\csc(\pi t)}{\pi}\Big|_{1/6}^{1/2}$

$$= \left[-\csc(\pi/2) + \csc(\pi/6)\right]1/\pi$$

$$= \left[-1 + 2\right]\frac{1}{\pi} = \boxed{1/\pi}$$

Second way:

$$\int_{1/6}^{1/2} \csc(\pi t)\cot(\pi t)\,dt = \int_{\pi/6}^{\pi/2} \csc(u)\cot(u)\,\frac{du}{\pi}$$

$t = 1/6 \Rightarrow u = \pi/6$
$t = 1/2 \Rightarrow u = \pi/2$

$$= -\frac{\csc(u)}{\pi}\Big|_{\pi/6}^{\frac{\pi}{2}} = \frac{1}{\pi}\left[-\csc\left(\frac{\pi}{2}\right) + \csc\left(\frac{\pi}{6}\right)\right] = \boxed{\frac{1}{\pi}}$$