Mat 135, Jan 10 2005

§5.4 Indefinite Integrals and the Net change Theorem.

You've now seen that to evaluate the definite integral

$$\int_a^b f(x)\,dx$$

we need to find an antiderivative of f. Once we've found an antiderivative, call it F, then by the fundamental theorem of Calculus,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b) - F(a)$$

New notation

$$\int f(x)\,dx = F(x) \quad \text{means} \quad F'(x) = f(x)$$

okay... now things get confusing. Look at the <u>definite</u> <u>integral</u> $\int_a^b f(x)dx$ this is a number. It depends on $a, b,$ and f but it does not depend on x.

Now we have this new animal, the <u>indefinite integral</u>, denoted

$$\int f(x)\,dx.$$

It's a function of x (and depends on f, of course). This takes some getting used to, but you'll manage.

For example,

$$\int 1\,dx = x + C$$

$$\int x\,dx = \frac{x^2}{2} + C$$

$$\int x^2\,dx = \frac{x^3}{3} + C$$

$$\int x^n\,dx = \begin{cases} \dfrac{x^{n+1}}{n+1} + C & n \neq -1 \\ \ln(|x|) + C & n = -1 \end{cases}$$

In all of these antiderivatives, c_1 is some constant. What c_1 you choose won't matter if you're using the indefinite integral (AKA antiderivative) in the second part of the fundamental theorem of calculus.

Basically, indefinite integrals are easy ways to write antiderivatives

$$\int \cos(x)\,dx = \sin(x) + c_1$$

$$\int \sin(x)\,dx = -\cos(x) + c_1$$

$$\int \sec^2(x)\,dx = \tan(x) + c_1$$

$$\int \csc^2(x)\,dx = -\cot(x) + c_1$$

$$\int \sec(x)\tan(x)\,dx = \sec(x) + c_1$$

$$\int \csc(x)\cot(x)\,dx = -\csc(x) + c_1$$

and so on.

note: $\int \frac{1}{x}\,dx = \ln(|x|) + C$ isn't the most general antiderivative. The more general antiderivative is

$$\begin{cases} \ln(x) + C & \text{if } x > 0 \\ \ln(|x|) + D & \text{if } x < 0 \end{cases}$$

But if you're using the indefinite integral to evaluate

$$\int_a^b \frac{1}{x}\,dx$$

then you need $\frac{1}{x}$ to be continuous on $[a,b]$ (if $a<b$) or on $[b,a]$ (if $a>b$). So you won't need the most general antiderivative:

$$\int_3^9 \frac{1}{x}\,dx = \ln(x) + C \Big|_3^9 \quad \text{the } C \text{ cancels}$$

$$\int_{-5}^{-2} \frac{1}{x}\,dx = \ln(|x|) + D \Big|_{-5}^{2} \quad \text{the } D \text{ cancels}$$

$$\int_{-2}^{1} \frac{1}{x}\,dx$$ DOESN'T MAKE SENSE AS A DEFINITE INTEGRAL

(Why? $\frac{1}{x}$ isn't continuous on $[-2, 1]$. And it's not continuous w/ a finite # of jump discontinuities. And it's not continuous w/ a finite # of removable discontinuities.)

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ex 6 Find the indefinite integral

$$\int \sqrt[3]{x}\,dx = \int x^{1/3}\,dx$$

$$= \ldots$$

want $x^{4/3}$ but

$\frac{d}{dx}\left(x^{4/3}\right) = \frac{4}{3} x^{1/3}$ off by $4/3$. So multiply by $3/4$.

$$\int \sqrt[3]{x}\,dx = \boxed{\frac{3}{4} x^{4/3} + C}$$

#16 find the indefinite integral

$$\int \cos(x) - 2\sin(x)\,dx = \int \cos(x)\,dx - \int 2\sin(x)\,dx$$

$$= \int \cos(x)\,dx - 2\int \sin(x)\,dx$$

$$= \sin(x) + C - 2(-\cos(x) + D)$$

$$= \sin(x) + 2\cos(x) + \tilde{C}$$

note: C is arbitrary and D is arbitrary. So really, $\tilde{C} = C - 2D$. But since C and D can take any value, we see that $\tilde{C}$ can take any value. So rather than writing $C - 2D$ above, I used a new constant $\tilde{C}$.

ex: $\displaystyle\int_1^3 1 + 2x - 4x^3\,dx = ?$

$$\int 1\,dx = x + C$$

$$\int x\,dx = \frac{x^2}{2} + \tilde{C}$$

$$\int x^3\,dx = \frac{x^4}{4} + D$$

$$\Rightarrow \int 1 + 2x - 4x^3\,dx = x + x^2 - x^4 + E$$

some constant E

$$\Rightarrow \int_1^3 1+2x-4x^3\,dx = x+x^2-x^4+E \Big|_1^3$$

$$= (3+9-81)-(1+1-1) = \boxed{-70}$$

ex 26

$$\int_1^2 \frac{y+5y^7}{y^3}\,dy = \int_1^2 \frac{y}{y^3}+\frac{5y^7}{y^3}\,dy$$

$$= \int_1^2 \frac{1}{y^2}+5y^4\,dy$$

know $\int \frac{1}{y^2}\,dy = -\frac{1}{y}+C$ & $\int y^4\,dy = \frac{y^5}{5}+D$

$\Rightarrow \int \frac{1}{y^2}+5y^4\,dy = -\frac{1}{y}+y^5+E$ some E

$$\int_1^2 \frac{1}{y^2}+5y^4\,dy = -\frac{1}{y}+y^5+E\Big|_1^2 = \left(-\frac{1}{2}+32\right)-\left(-\frac{1}{1}+1\right)$$

$$= \boxed{63/2}$$

ex 38

$$\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$$

$$= \int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx$$

$$= \int_4^9 x + 2 + \frac{1}{x}\, dx$$

$$= \left. \frac{x^2}{2} + 2x + \ln(x) + C \right|_4^9$$

$$= \left(\frac{81}{2} + 18 + \ln(9)\right) - \left(\frac{16}{2} + 8 + \ln(4)\right)$$

$$= \boxed{\frac{85}{2} + \ln(9) - \ln(4)}$$

ex 54 The velocity of a particle moving along a line is

$$v(t) = t^2 - 2t - 8 \quad m/sec$$

for $1 \le t \le 6$. Find the distance travelled by the particle in the given time interval.

recall that $s(t)$ = particle position has

$\frac{ds}{dt} = v(t)$. That is, $s(t)$ is an antiderivative of $v(t)$.

$$\Rightarrow \int v(t)\,dt = s(t)$$

$$\Rightarrow \int t^2 - 2t - 8\,dt = s(t)$$

$$\Rightarrow \frac{t^3}{3} - t^2 - 8t + C_1 = s(t)$$

position at $t = 1$

distance travelled = $s(6) - s(1)$

position at $t = 6$

$$= \frac{t^3}{3} - t^2 - 8t + C_1 \Big|_1^6 = \boxed{-\frac{10}{3}}$$