Feb 4, 2005

§7.8 Improper Integrals.

Recall that <u>definite</u> integrals

$$\int_a^b f(x)\,dx$$

were defined when f is continuous or has finitely many removeable or jump discontinuities on $[a,b]$.

<u>Note:</u> $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$, $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$, $\displaystyle\int_{-1}^1 \frac{1}{x^{1/3}}\,dx$

are <u>not</u> definite integrals.

In the first case, $\frac{1}{x^2}$ is continuous on $[1,\infty)$ but definite integrals require this on $[a,b]$. (which implies $a \neq -\infty$ and $b \neq \infty$.)

In the second two cases, $f(x)$ has a blow-up at $x=0$ so $[0,1]$ and $[-1,1]$ fail too.

Specifically,

$\int_0^1 \frac{1}{\sqrt{x}} dx$ is not a definite integral

because $f(x)$ has a blow-up / infinite discontinuity at $x=0$ so we don't have the required good behaviour on the closed interval $[0,1]$.

Similarly, for the case $\int_{-1}^{1} \frac{1}{x^{1/3}} dx$ we don't have the required good behaviour of $f(x)$ on $[-1,1]$.

We can, however define <u>improper</u> integrals.

(1) $\int_1^{\infty} \frac{1}{x^2} dx = \lim_{R \to \infty} \int_1^R \frac{1}{x^2} dx$ if this limit exists

(2) $\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \to 0+} \int_t^1 \frac{1}{\sqrt{x}} dx$ if this limit exists

(3) $\int_{-1}^{1} \frac{1}{x^{1/3}} dx = \lim_{t \to 0-} \int_{-1}^{t} \frac{1}{x^{1/3}} dx + \lim_{t \to 0+} \int_t^1 \frac{1}{x^{1/3}} dx$

Remember

limits existing also means that the limit is finite!

$\lim_{x \to 0} \frac{1}{x^2}$ does not exist.

even though we then write $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R \to \infty} \int_1^R \frac{1}{x^2}\,dx = \lim_{R \to \infty} \left. -\frac{1}{x} \right|_1^R$$

$$= \lim_{R \to \infty} -\frac{1}{R} + \frac{1}{1} = \boxed{1}$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0+} \int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim \left. 2\sqrt{x} \right|_t^1$$

$$= \lim_{t \to 0+} 2 - 2\sqrt{t} = \boxed{2}$$

$$\int_{-1}^{1} \frac{1}{x^{1/3}}\,dx = \lim_{t \to 0^-} \int_{-1}^{t} \frac{1}{x^{1/3}}\,dx + \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x^{1/3}}\,dx$$

$$= \lim_{t \to 0^-} \frac{3}{2} x^{2/3} \Big|_{-1}^{t} + \lim_{t \to 0^+} \frac{3}{2} x^{2/3} \Big|_{t}^{1}$$

$$= \lim_{t \to 0^-} \left(\frac{3}{2} t^{2/3} - \frac{3}{2}\right) + \lim_{t \to 0^+} \left(\frac{3}{2} - \frac{3}{2} t^{2/3}\right)$$

$$= \boxed{0}$$

ex:

$$\int_{1}^{\infty} \frac{1}{x}\,dx = \lim_{R \to \infty} \int_{1}^{R} \frac{1}{x}\,dx$$

$$= \lim_{R \to \infty} \ln(x) \Big|_{1}^{R} = \lim_{R \to \infty} \ln(R) - \ln(1)$$

does not exist!

$$\int_{0}^{1} \frac{1}{x}\,dx = \lim_{t \to 0^+} \int_{t}^{1} \frac{1}{x}\,dx = \lim_{t \to 0^+} \ln(1) - \ln(t)$$

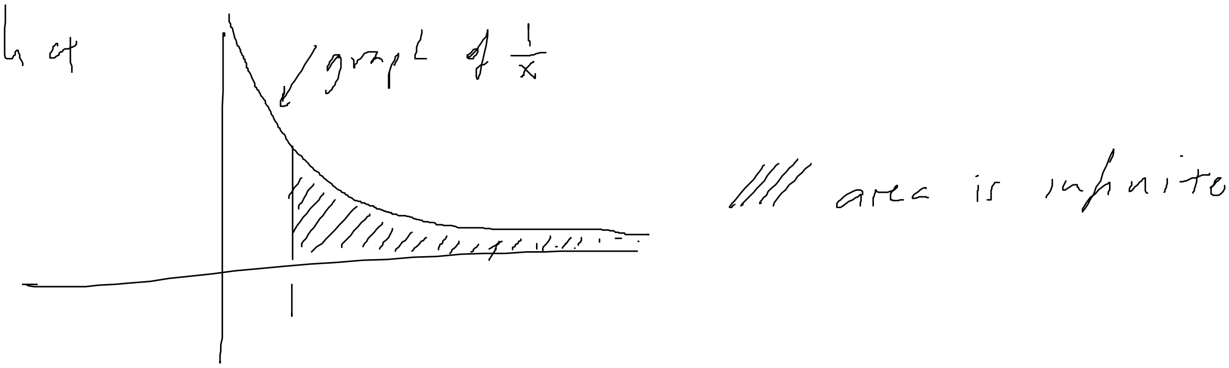
does not exist!

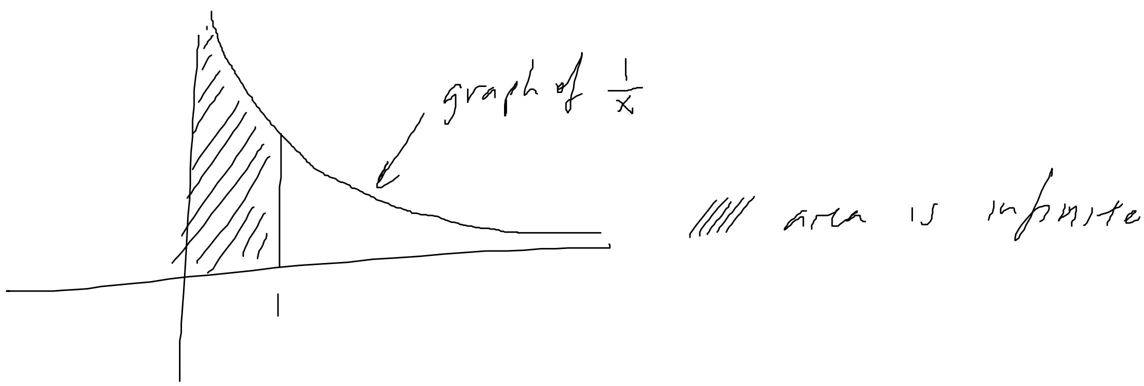
Actually, we already knew that!

Since $\lim_{R \to \infty} \int_1^R \frac{1}{x} dx = \lim_{R \to \infty} \ln(R)$

and $\lim_{t \to 0+} \int_t^1 \frac{1}{x} dx = \lim_{t \to 0+} \ln(t)$

And from our class on logarithms, we already knew that





In general, some decays to infinity are fast enough to ensure a finite area (we saw $1/x^2$ is) and some are too slow (we saw $1/x$ is too slow)

What rates work?

$$\int_7^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty}\int_7^R \frac{1}{x^p}\,dx$$

$$= \begin{cases} \lim\limits_{R\to\infty} \frac{1}{1-p} \, x^{1-p}\Big|_7^R & \text{if } p \neq 1 \\ \lim\limits_{R\to\infty} \ln(x)\Big|_7^R & \text{if } p = 1 \end{cases}$$

$$= \lim_{R\to\infty} \begin{cases} \frac{1}{1-p}\left(R^{1-p} - 7^{1-p}\right) & \text{if } p \neq 1 \\ \ln(R) - \ln(7) & \text{if } p = 1 \end{cases}$$

we know $\ln(R) \to \infty$ as $R \to \infty$ (we proved this already.

if $1-p > 0$ then $R^{1-p} \to \infty$ as $R \to \infty$.

if $1-p < 0$ then $R^{1-p} \to 0$ as $R \to \infty$

conclusion: $\int_7^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$

note: 7 didn't matter. Any lower bound a w/ $a > 0$ would do.

Similarly, some blow-ups are fast enough that the area is finite and others are too slow.

$$\int_2^3 \frac{1}{(x-2)^p}\,dx = \lim_{t\to 2^+} \int_t^3 \frac{1}{(x-2)^p}\,dx$$

$$= \lim_{t\to 2^+} \begin{cases} \frac{1}{1-p}(x-2)^{1-p}\Big|_t^3 & \text{if } p \neq 1 \\ \ln(x-2)\Big|_t^3 & \text{if } p = 1 \end{cases}$$

$$= \lim_{t\to 2^+} \begin{cases} \frac{1}{1-p}\left(1^{1-p} - (t-2)^{1-p}\right) & \text{if } p \neq 1 \\ \ln(1) - \ln(t-2) & \text{if } p = 1 \end{cases}$$

We know that $\ln(t-2) \to -\infty$ as $t \to 2^+$ so the $p=1$ integral is divergent.

If $1-p > 0$ then $(t-2)^{1-p} \to 0$ as $t\to 2^+$ so the integral converges. If $p-1<0$ then $(t-2)^{1-p} \to \infty$ as $t \to 2^+$ and so the integral diverges.

<u>conclusion:</u>

$$\int_a^b \frac{1}{(x-a)^p}\,dx \quad \text{where } b<\infty \text{ is}$$

convergent if $p<1$ and divergent if $p\geq 1$.

<u>ex:</u>

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{R\to\infty}\int_2^R \frac{1}{x\ln(x)}\,dx$$

$$= \lim_{R\to\infty} \ln(\ln(x))\Big|_2^R$$

$$= \lim_{R\to\infty} \ln(\ln(R)) - \ln(\ln(2))$$

<u>divergent integral</u>

So the speed-up provided by $\ln(x)$ didn't help.
Transl: $x\ln(x)\to\infty$ faster than $x\to\infty$. So $\frac{1}{x\ln(x)}\to 0$ faster than $\frac{1}{x}\to 0$. But $\frac{1}{x\ln(x)}$ doesn't go to zero fast

enough to ensure that

$$\int_2^\infty \frac{1}{x\ln(x)}\,dx \quad \text{converges.}$$

Integral comparison test.

assume f and g are continuous functions on $[a,\infty)$ and $f(x) \geq g(x) \geq 0$ for all $x \geq a$.

1. If $\int_a^\infty g(x)\,dx$ diverges then $\int_a^\infty f(x)\,dx$ must diverge.

2. If $\int_a^\infty f(x)\,dx$ converges then $\int_a^\infty g(x)\,dx$ must also converge.

Similarly, there's an integral comparison test for functions that have blow-ups

ex! 11

$$\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx = \lim_{R\to\infty} \int_{-R}^R \frac{x}{1+x^2}\,dx$$

$$= \lim_{R\to\infty} \frac{1}{2}\ln(1+x^2)\Big|_{-R}^{R} \quad \underline{\text{divergent}}$$

ex 20 $\int_{-\infty}^{6} re^{r/3}\,dr = \lim_{R\to-\infty} \int_{R}^{6} re^{r/3}\,dr$

$= \lim_{R\to-\infty} 3re^{r/3} - 9e^{r/3}\Big|_{R}^{6}$

$= \lim_{R\to-\infty} 18e^{2} - 9e^{2} - 3Re^{R/3} + 9e^{R/3}$

we know $e^{R/3} \to 0$ as $R \to -\infty$

and $Re^{R/3} \to 0$ as $R \to -\infty$ so

$\int_{-\infty}^{6} re^{r/3}\,dr = \boxed{18e^{2} - 9e^{2}}$

ex 34: $\int_{0}^{1} \frac{1}{4y-1}\,dy = \lim_{t\to 1/4^-} \int_{0}^{t} \frac{1}{4y-1}\,dy + \lim_{t\to 1/4^+} \int_{t}^{1} \frac{1}{4y-1}\,dy$

$= \lim_{t\to 1/4^-} \frac{1}{4} \ln|4y-1|\Big|_{0}^{t} + \lim_{t\to 1/4^+} \ln(4y-1)\Big|_{t}^{1}$

$= \lim_{t\to 1/4^-} \frac{1}{4}\ln|4t-1| - \frac{1}{4}\ln(1) + \lim_{t\to 1/4^+} \frac{1}{4}\ln(3) - \frac{1}{4}\ln(4t-1)$

$\boxed{\text{diverges}}$

50 $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ diverges

why? $$\underbrace{\frac{2+e^{-x}}{x}}_{f(x)} \geq \underbrace{\frac{1}{x}}_{g(x)} \geq 0$$

and since $\displaystyle\int_1^\infty g(x)\,dx$ diverges, we know $\displaystyle\int_1^\infty f(x)$ does too.

52 $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ converges.

why? $$0 \leq \overbrace{\frac{x}{\sqrt{1+x^6}}}^{g(x)} \leq \overbrace{\frac{1}{x^2}}^{f(x)}$$

and since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, we know $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ conv.

#54 $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ converges.

why? for x in $[0,1]$,

$0 \le e^{-x} \le 1$

$\Rightarrow$

$$0 \le \underbrace{\frac{e^{-x}}{\sqrt{x}}}_{g(x)} \le \underbrace{\frac{1}{\sqrt{x}}}_{f(x)}.$$

Since $\int_0^1 f(x)\,dx$ converges, $\int_0^1 g(x)$ converges too.

$\displaystyle\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ converges too.

why? for x in $[1,\infty)$,

$$\frac{1}{\sqrt{x}} \le 1 \Rightarrow 0 \le \underbrace{\frac{e^{-x}}{\sqrt{x}}}_{g(x)} \le \underbrace{e^{-x}}_{f(x)}$$

Since $\int_1^\infty f(x)\,dx$ converges, $\int_1^\infty g(x)$ does too.