Mat 135 Feb 28, 2005

§ 9.7 Predator-Prey systems.

Up until now, we've considered a single differential equation that models a single thing (fish population or money owed or...)

Now we will consider two interacting populations: rabbits and wolves. Wolves eat rabbits, so the more wolves there are, the faster the rabbits will die. Similarly, the more rabbits there are to eat the more wolf cubs can be born!

$R(t) =$ # of rabbits at time t

$W(t) =$ # of wolves at time t.

if there were no wolves ($W(t) = 0$ for all t) then the rabbits would happily live off the land and breed like rabbits:

$$\frac{dR}{dt} = kR \quad \text{where } k > 0$$

If there were no rabbits to eat, then the wolves would die out:

$$\frac{dW}{dt} = -rW \quad \text{where } r > 0.$$

What if you have some rabbits & wolves living in the same place?

$$\frac{dR}{dt} = kR - aRW \qquad \text{where } k, a > 0$$

from how the rabbits interact with the environment

how the wolves eat the rabbits Note that the more wolves there are, the more rabbits get eaten. And the more rabbits there are the more likely one of them will run into a wolf and get eaten

$$\frac{dW}{dt} = -rW + bRW$$

from how the wolves interact with the environment

how the presence of rabbits increases the birthrate. The more wolves there are to breed, the more cubs there will be. And the more rabbits there are to eat the more cubs there will be.

So for the wolf-rabbit population, we have two differential equations that have to be solved simultaneously:

$$\begin{cases} \dfrac{dR}{dt} = kR - aRW \\ \dfrac{dW}{dt} = -rW + bRW \end{cases}$$

with initial data $R(0) = R_0$ and $W(0) = W_0$. The four constants $k, a, r,$ and b are estimated by biologists from field measurements. Whether it's rabbits + wolves or sharks + seals or birds + frogs would show up in the values of the constants.

Fact: there is one constant solution where $R(t) = R_0$ for all time and $W(t) = W_0$ for all time. To find it we seek

$$\frac{dR}{dt} = 0 \quad \text{(since constant in time)}$$

$$\| $$

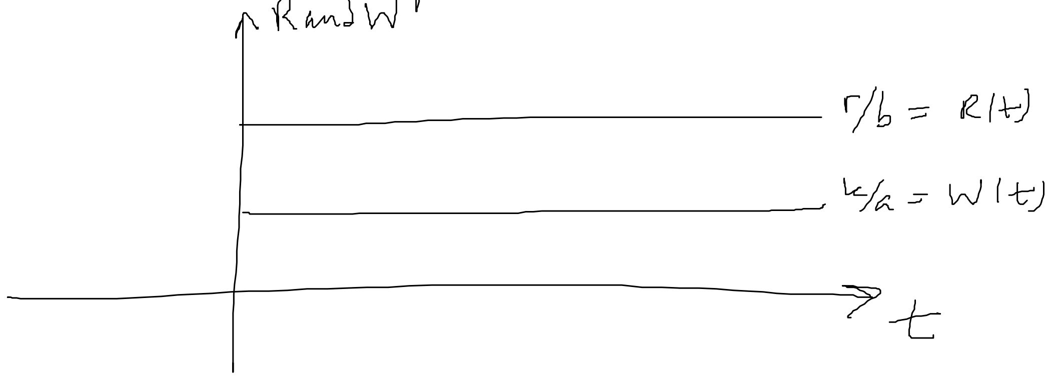
$$kR_0 - aR_0W_0 = R_0(k - aW_0) \Rightarrow W_0 = k/a$$

$$\frac{dW}{dt} = 0 \quad \text{(since constant in time}$$

$$\| $$

$$-rW_0 + bR_0W_0 = W_0(-r + bR_0) \Rightarrow R_0 = r/b$$

So if you start with r/b rabbits and k/a wolves then you will always have r/b rabbits and k/a wolves.



What if you don't start with this very special set of initial conditions?

$$\frac{dR}{dt} = kR - aRW = R(k - aW)$$

when is $\frac{dR}{dt} = 0$? > 0? < 0?

First, we'll assume there are some rabbits around so $R(t_0) > 0$. Then we see that

$$W(t_0) < k/a \Rightarrow \frac{dR}{dt}(t_0) > 0$$

$$W(t_0) = k/a \Rightarrow \frac{dR}{dt}(t_0) = 0$$

$$W(t_0) > k/a \Rightarrow \frac{dR}{dt}(t_0) < 0$$

Look at the $\frac{dW}{dt}$ equation

$$\frac{dW}{dt}(t_0) = -rW(t_0) + bR(t_0)W(t_0)$$
$$= W(t_0)\left(bR(t_0) - r\right)$$

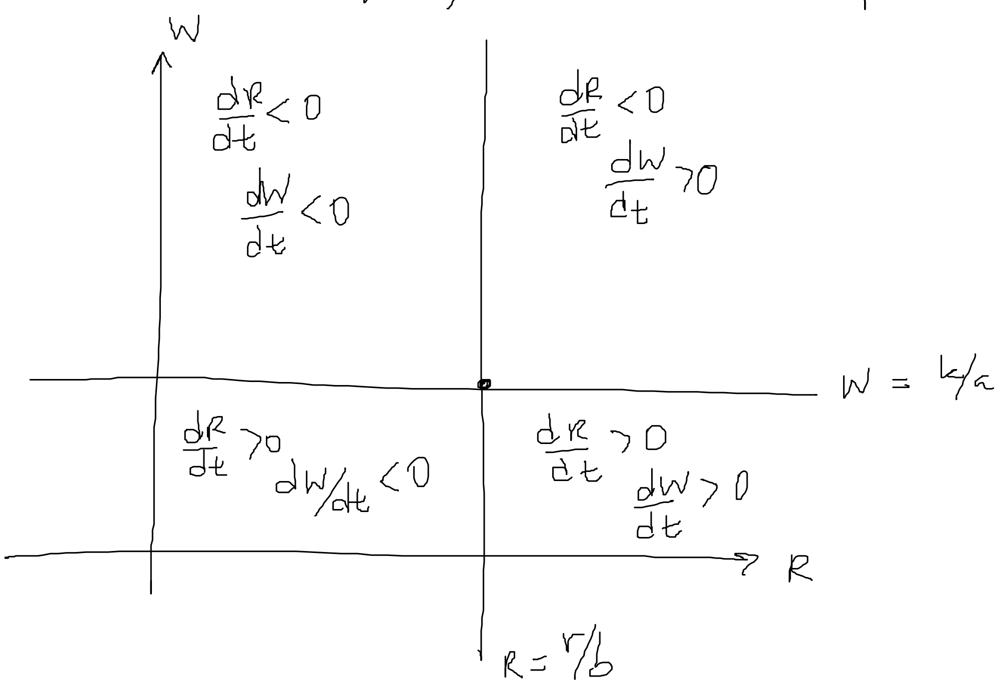
again, we'll assume there are some wolves around, so $W(t_0) > 0$. Then we see that

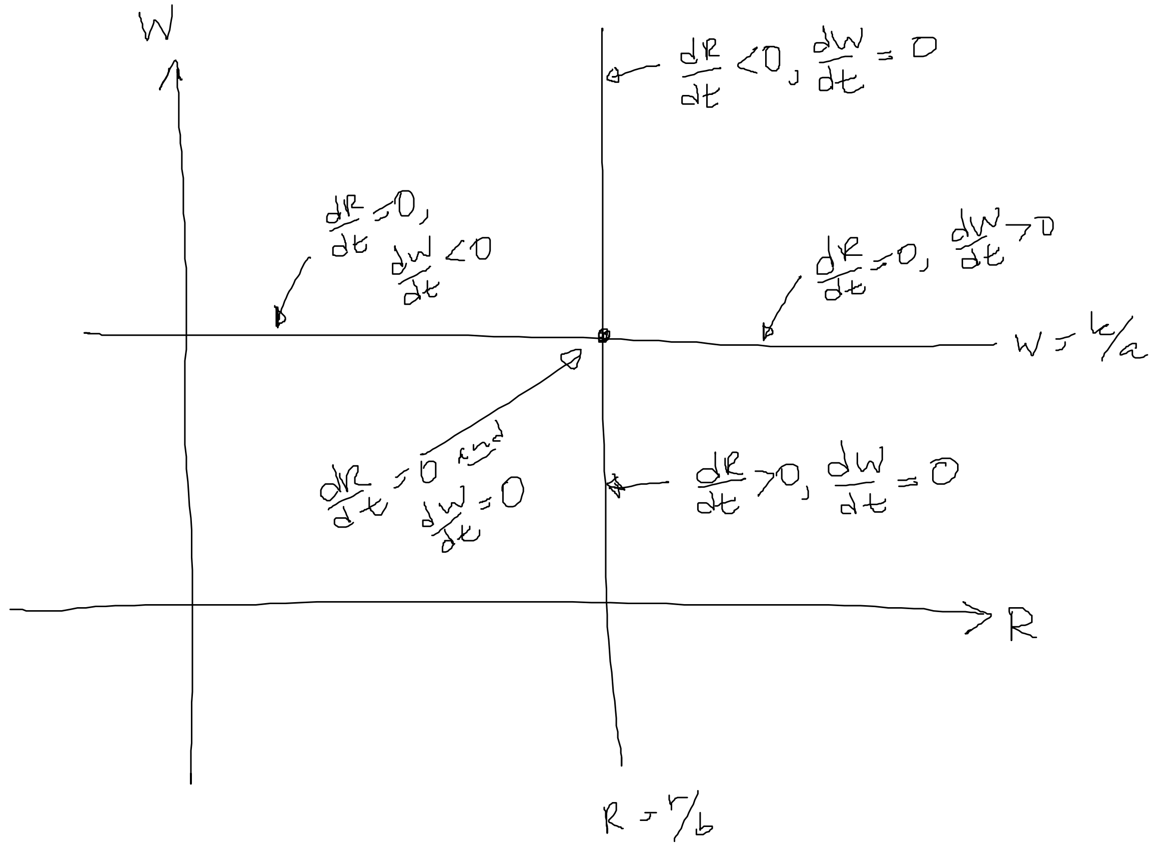
$$R(t_0) > r/b \Rightarrow \frac{dW}{dt}(t_0) > 0$$

$$R(t_0) = r/b \Rightarrow \frac{dW}{dt}(t_0) = 0$$

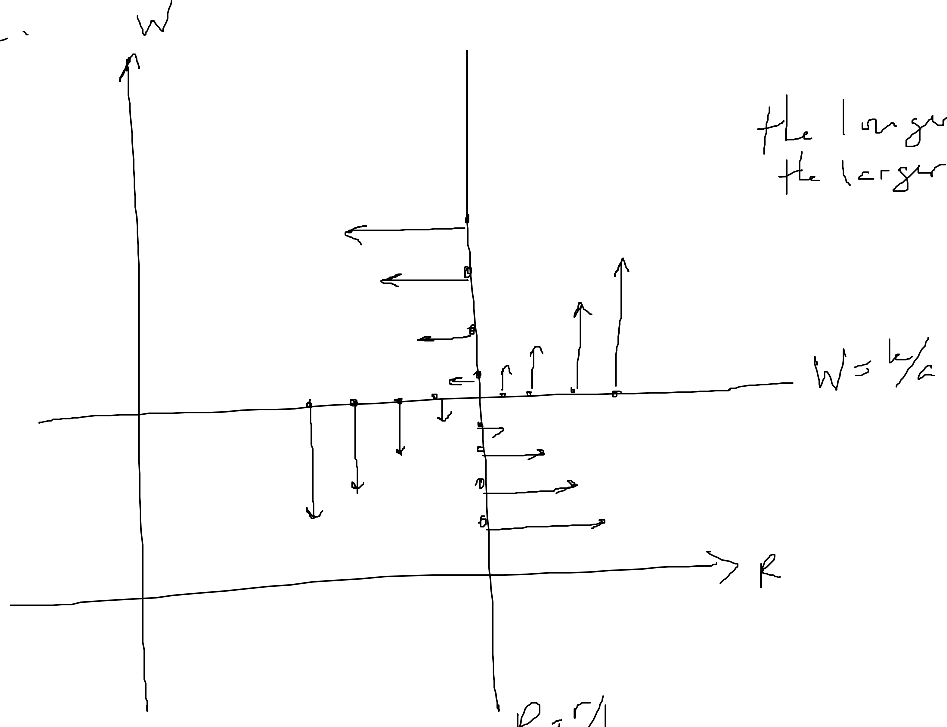
$$R(t_0) < r/b \Rightarrow \frac{dW}{dt}(t_0) < 0$$

We combine all this information in a phase-plane. You haven't seen this before, it's the RW plane

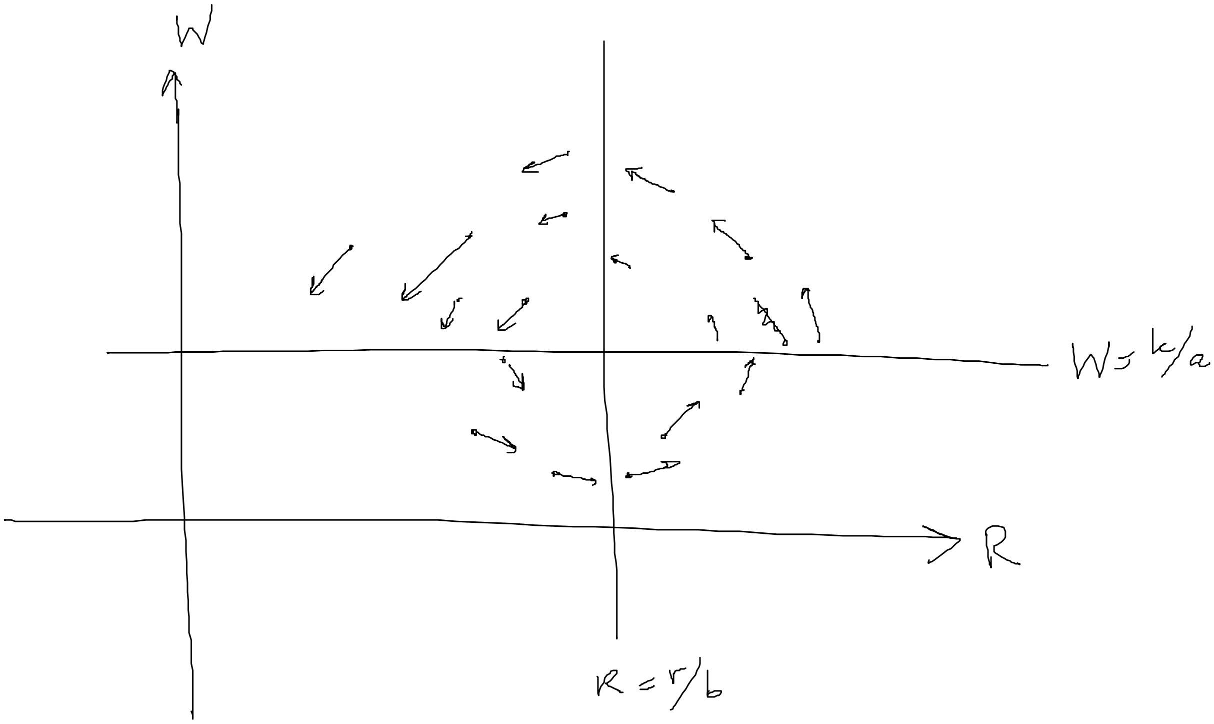




Now I'll plot some initial points and will give arrows that point in the direction the points will move.

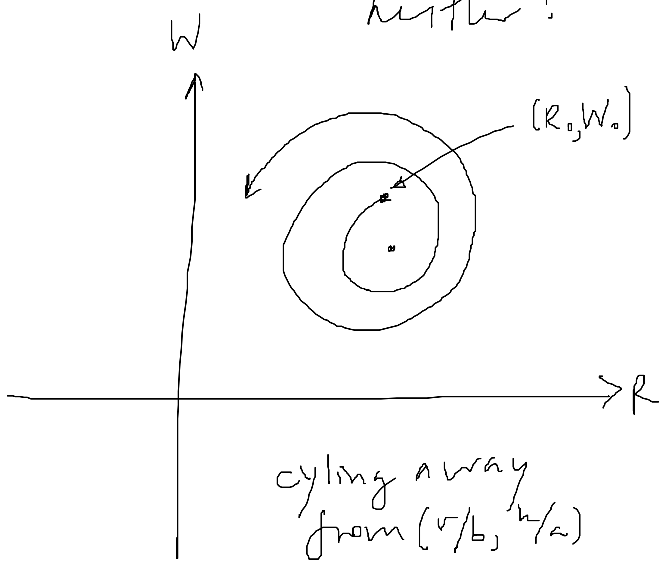


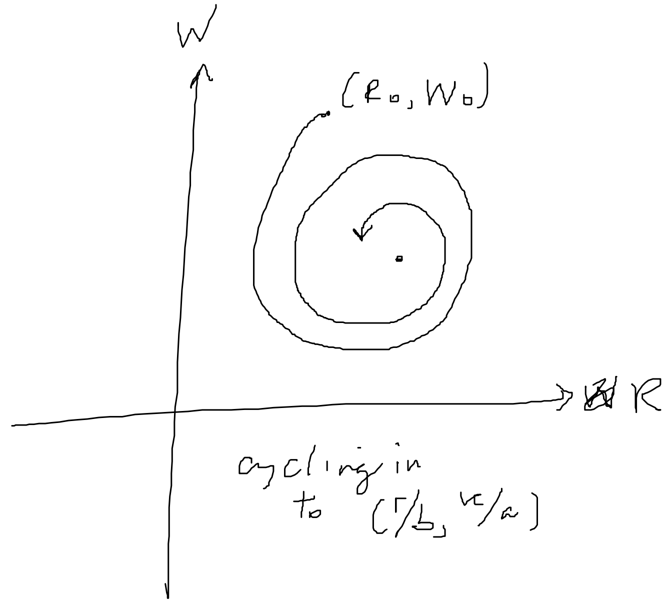
the longer the arrow, the larger the velocity



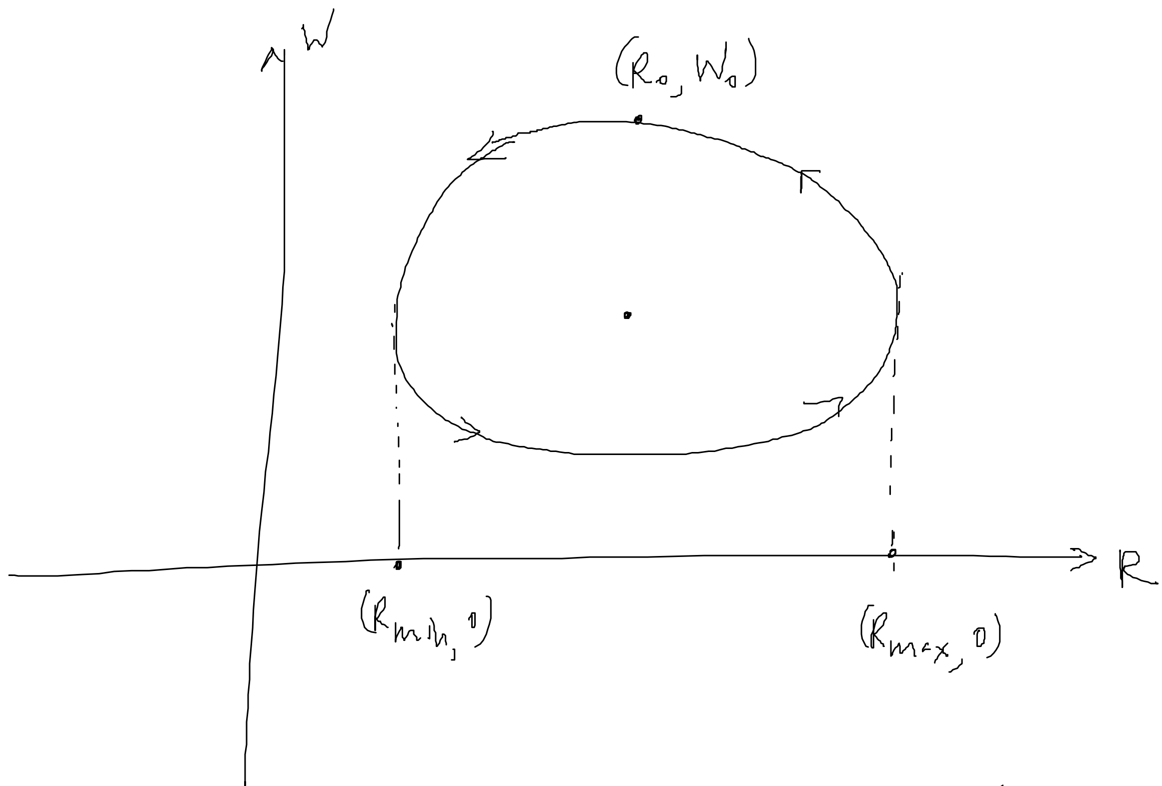
putting it all together, it looks like the solution $(R(t), W(t))$ is moving counter-clockwise around the equilibrium point $(r/b, k/a)$.

Is it cycling into the equilibrium point?
away from the eq. point?
neither?





In fact, it turns out that it's neither!

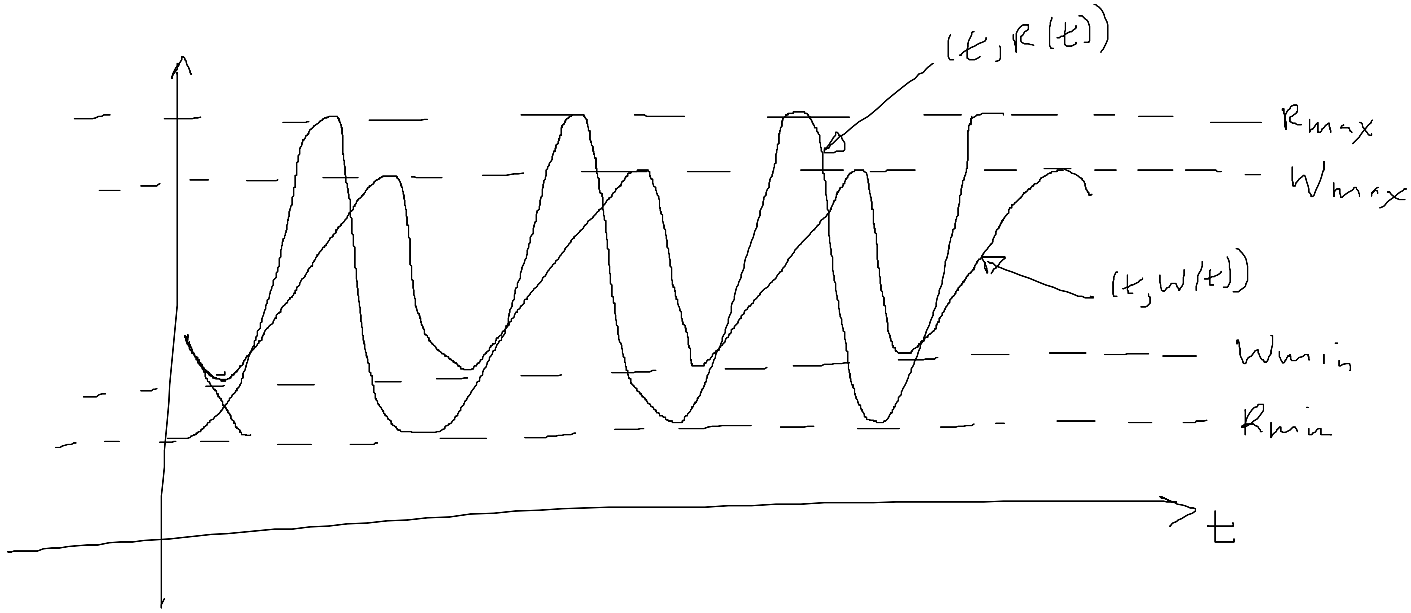


you see that there's R_{max} and R_{min} the maximum and minimum rabbit populations and these values are achieved infinitely many times.

Similarly, there's a maximum and minimum wolf population W_{max} and W_{min}

(R_{max}, R_{min}, W_{max}, and W_{min} depend on the initial data R_0, W_0 as well as the constants r, a, k, and b.)

try plotting R(t) and W(t) as functions of t to better understand:



R(t) and W(t) are periodic. They aren't as simple as cos(t) and sin(t) but they're periodic nonetheless

Please go to

http://www.aw-bc.com/ide/Media/JavaTools/popltkvl.html

and play around!