Mat 135 Feb 25, 2005

§ 9.5 The Logistic Equation

consider the differential equation w/ initial data

$$\begin{cases} \dfrac{dP}{dt} = t \\ P(0) = P_0 \end{cases}$$

We know that for $t>0$ $\frac{dP}{dt}>0$. So whatever the solution $P(t)$ is, it's increasing.

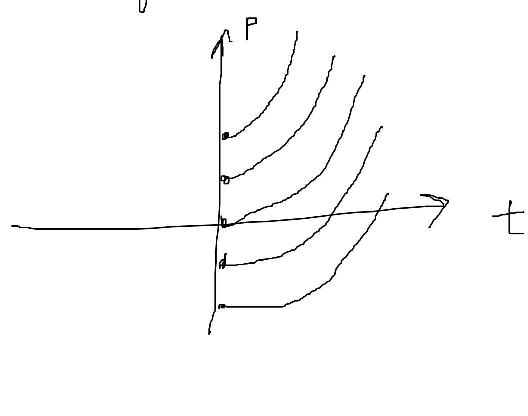
$$\frac{d}{dt}\frac{dP}{dt} = \frac{d^2P}{dt^2}$$

$$\| $$

$$\frac{d}{dt}(t) = 1$$

This shows us that $\frac{d^2P}{dt^2}>0$ so that the graph of $P(t)$ is concave up.

In fact, the solution is $P(t) = P_0 + \frac{t^2}{2}$ and it satisfies what we expected: increasing for $t>0$, & concave up.



Consider the equation & initial data

$$\begin{cases} \dfrac{dP}{dt} = kP & \text{with } k > 0 \\ P(0) = P_0 \end{cases}$$

if $P(t_0) > 0$ then $\frac{dP}{dt}(t_0) > 0$ and so $P(t)$ will continue to increase as t passes through t_0.

If $P(t_0) < 0$ then $\frac{dP}{dt}(t_0) < 0$ and $P(t)$ will decrease as t passes through t_0.

Conclusion: If the initial data $P_0 > 0$ then the solution $P(t)$ will be increasing for all $t > 0$.
If $P_0 < 0$ then $P(t)$ will be decreasing for all $t > 0$

What about the concavity of $P(t)$?

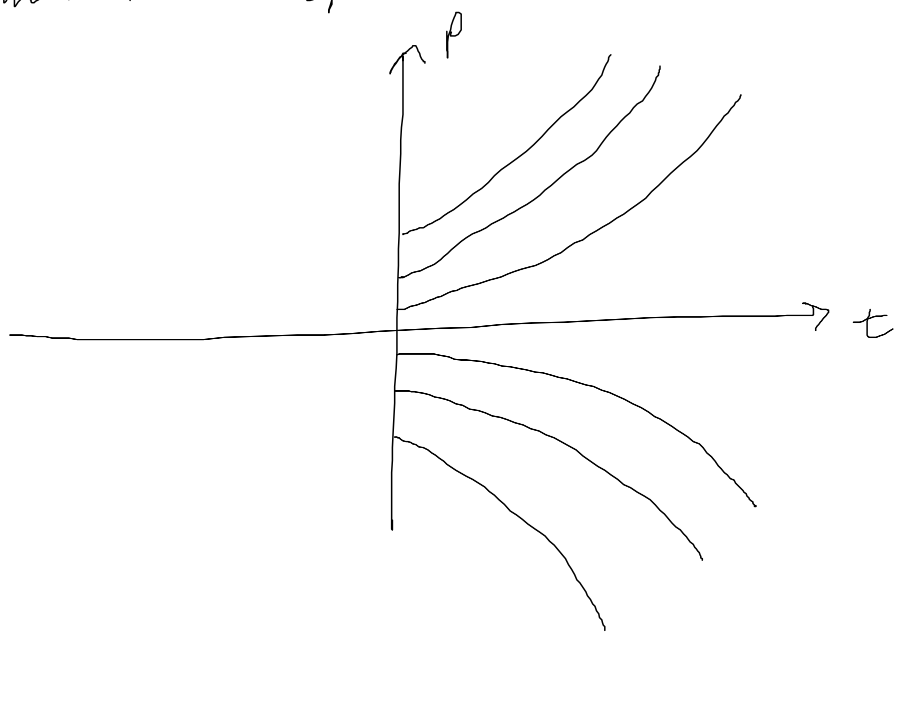
$$\frac{d^2P}{dt^2} = \frac{d}{dt}\frac{dP}{dt} = \frac{d}{dt}(kP) = k\frac{dP}{dt} = k(kP) = k^2P$$

So we see that if $P(t) > 0$ then $\frac{d^2P}{dt^2} > 0$

and if $P(t) < 0$ then $\frac{d^2P}{dt^2} < 0$

Conclusion: if $P_0 > 0$ then the solution $P(t)$ has a concave-up graph. If $P_0 < 0$ then the solution $P(t)$ has a concave-down graph.

In fact, we know the solution $P(t) = P_0 e^{kt}$ and it satisfies both conclusions!



Now, consider the logistic equation with initial data P_0:

$$\begin{cases} \frac{dP}{dt} = kP(1 - P/K_1) & \text{where } k, K_1 > 0 \\ P(0) = P_0 \end{cases}$$

note that
$$P(t) < 0 \Rightarrow \frac{dP}{dt} < 0$$
$$P(t) = 0 \Rightarrow \frac{dP}{dt} = 0$$
$$0 < P(t) < K_1 \Rightarrow \frac{dP}{dt} > 0$$
$$P(t) = K_1 \Rightarrow \frac{dP}{dt} = 0$$
$$P(t) > K_1 \Rightarrow \frac{dP}{dt} < 0$$

conclusion: if $P_0 < 0$ then the solution $P(t)$ will decrease as t increases. If $0 < P_0 < K_1$ then the solution $P(t)$ will increase as t increases. If $P_0 > K_1$ then the solution $P(t)$ will decrease as t increases. If $P_0 = 0$ then $P(t) = 0$ for all t. If $P_0 = K_1$ then $P(t) = K_1$ for all t.

Now let's check out the concavity of the graph of $P(t)$.

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\frac{dP}{dt} = \frac{d}{dt}\left(kP\left(1 - P/K_1\right)\right)$$

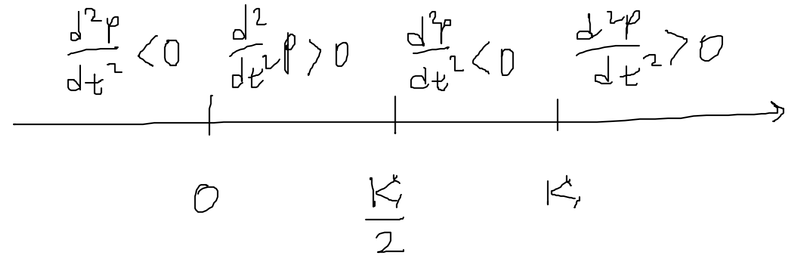
$$= k\frac{dP}{dt}\left(1 - P/K_1\right) + kP\left(-\frac{1}{K_1}\frac{dP}{dt}\right)$$

$$= \frac{dP}{dt}\left[k - \frac{kP}{K_1} - \frac{kP}{K_1}\right]$$

$$= \frac{dP}{dt}\, k\left[1 - 2P/K_1\right]$$

$$= k^2 P\left(1 - P/K_1\right)\left(1 - 2P/K_1\right)$$

So

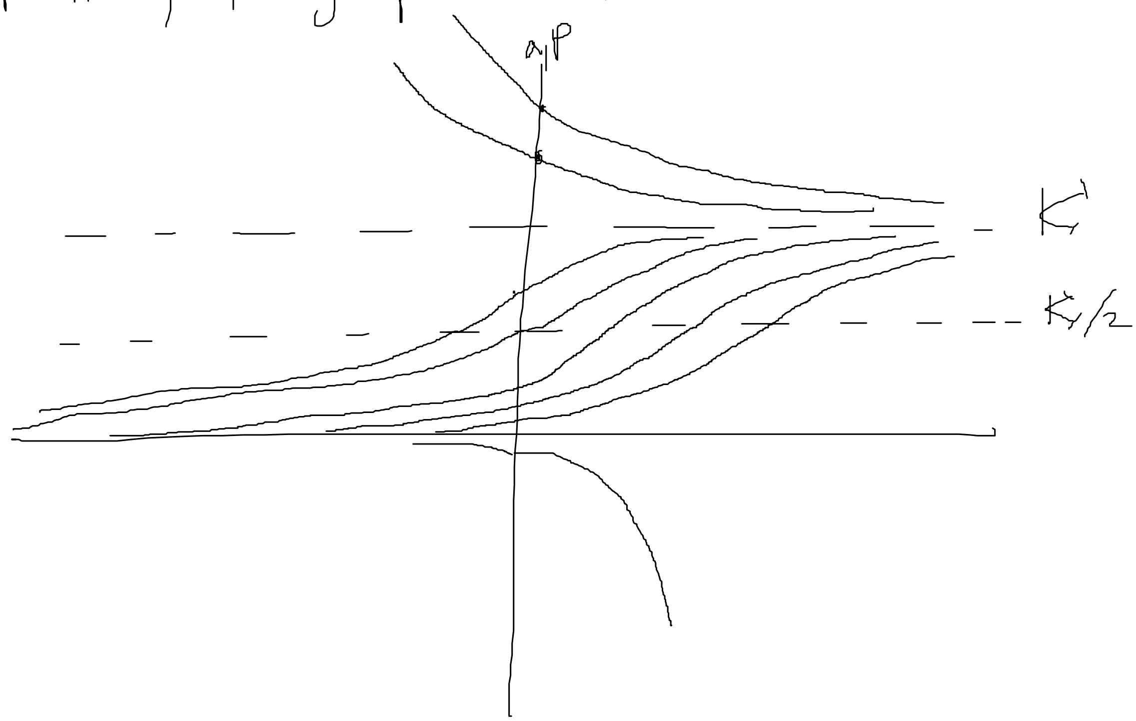


Conclusion: when $P(t)$ passes through the value $K_1/2$ the graph changes from concave up to concave down. The graph is concave up for $P(t) > K_1$ and concave down for $P(t) < 0$.

We found the solution before, since the equation is separable:

$$P(t) = \frac{P_0 K_1^* e^{kt}}{K_1^* - P_0 + P_0 e^{kt}}$$

Plotting the graph one finds the solutions look like



which is what we expected.

Note that if $P_0 < 0$ then the denominator can equal zero. Since $K_1^* - P_0 + P_0 e^{kt} = 0$ when $t = \frac{1}{k} \ln\left(1 - \frac{K_1^*}{P_0}\right)$

we see that if $P_0 < 0$ then the solution $P(t)$ goes to $-\infty$ as t increases from 0 to $\frac{1}{k} \ln\left(1 - \frac{K_1^*}{P_0}\right)$

On the other hand, if $P_0 > 0$ then the denominator $K_1 - P_0 + P_0 e^{kt}$ will never equal zero. So $P(t)$ will be a nice function for all t. And we can ask what $P(t)$ does as $t \to \infty$ in this case

$$\text{if } P_0 > 0 \text{ then } \lim_{t\to\infty} P(t) = K_1$$

i.e whatever the initial population is, if it's positive then the entire population will get closer & closer to the carrying capacity K_1.

If $P_0 > K_1$ the population will decrease towards K_1. If $0 < P_0 < K_1$ it will increase towards K_1.

#3 The pacific halibut fishery has been modelled by the differential equation

$$\frac{dy}{dt} = ky\left(1 - y/K_1\right)$$

where $y(t)$ is the biomass in kg at time t, $K_1 = 8 \times 10^7$ kg and $k = .71$/year

a) If $y(0) = 2\times 10^7$ kg, find the biomass a year later

$$y(t) = \frac{(2\times10^7)(8\times10^7)\,e^{.71t}}{8\times10^7 - 2\times10^7 + (2\times10^7)e^{.71t}}\ \text{kg}$$

$$= \frac{16\times10^{14}\,e^{.71t}}{6\times10^7 + 2\times10^7 e^{.71t}}\ \text{kg}$$

$$= \frac{8\times10^7\,e^{.71t}}{3 + e^{.71t}}\ \text{kg}$$

$$\boxed{y(1) = \frac{8\times10^7\,e^{.71}}{3 + e^{.71}}\ \text{kg}} \cong 3.23\times10^7\ \text{kg}$$

b) How long will it take the biomass to reach 4×10^7 kg?

$$y(t_0) = 4\times10^7 = \frac{8\times10^7\,e^{.71t_0}}{3 + e^{.71t_0}}$$

$$1 = \frac{2e^{t_0}}{3 + e^{.71t_0}} \Rightarrow 3 + e^{.71t_0} = 2e^{.71t_0}$$

$$\Rightarrow \boxed{t_0 = \frac{\ln(3)}{.71}} \cong 1.55\ \text{yr.}$$

#8 Biologists stocked a lake w/ 400 fish and estimated the carrying capacity to be 10,000 fish. The # of fish tripled in the first three years.

a) Assuming the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

$$P(t) = \frac{(400)(10000)e^{kt}}{10000-400 + 400e^{kt}}$$

$$= \frac{4000000e^{kt}}{9600 + 400e^{kt}}$$

$$= \frac{10000e^{kt}}{24 + e^{kt}}$$

What's k? Know $P(3) = 3 \cdot P(0) = 3(400) = 1200$

$$\Rightarrow \quad 1200 = \frac{10000e^{3k}}{24 + e^{3k}} \Rightarrow k = \frac{1}{3}\ln\left(\frac{36}{11}\right)$$

b) How long will it take for the population to reach 5000? $5000 = P(t_0) = \dfrac{1000e^{\frac{1}{3}\ln(36/11)t_0}}{24 + e^{1/3\ln(36/11)t_0}}$

$$\Rightarrow t_0 = \frac{3\ln(24)}{\ln(36/11)} \cong 8.04 \text{ years}$$

#11 modify the logistic equation to reflect fishing:

$$\frac{dp}{dt} = .08p\left(1 - p/1000\right) - 15$$

fishing

then $$\int \frac{dp}{.08p(1-p/1000)-15} = \int dt$$

$$\Rightarrow 25\ln(p-250) - 25\ln(p-750) = t + C_1$$

$$\Rightarrow 25\ln\left(\frac{p-250}{p-750}\right) = t + C$$

$$\Rightarrow \ln\left(\frac{p-250}{p-750}\right) = \frac{t}{25} + C$$

$$\Rightarrow \frac{p-250}{p-750} = Ce^{t/25}$$

$$\Rightarrow p = \frac{250(1-3Ce^{t/25})}{1-Ce^{t/25}}$$

$$P(0) = P_0 = \frac{250(1-3C)}{1-C} \Rightarrow C = \frac{P_0 - 250}{P_0 - 750}$$

$$\Rightarrow P(t) = \frac{250\left((P_0-750) - 3(P_0-250)e^{t/25}\right)}{P_0 - 750 - (P_0-250)e^{t/25}}$$

What are the equilibrium solutions? i.e. when does $\frac{dP}{dt} = 0$? $\frac{dP}{dt} = 0 \Leftrightarrow .08P(1 - P/1000) - 15 = 0$

$$\Leftrightarrow P = 250 \text{ or } 750$$

Graph the solutions with $P_0 = 200$ and $P_0 = 300$

$P_0 = 200 \Rightarrow P(t) = \frac{250(11 - 3e^{t/25})}{11 - e^{t/25}}$ ← Note these fish die out $P(t_0) = 0$ when $t_0 = 25\ln\left(\frac{11}{3}\right)$

$P_0 = 300 \Rightarrow P(t) = \frac{750(3 + e^{t/25})}{9 + e^{t/25}}$ ← these fish keep going and as $t \to \infty$ $P(t) \to 750$