Mat 135 Feb 23, 2005

§9.4 Exponential growth & decay.

Population growth/decay.

$P(t)$ = # of rabbits at time t.

$P(t+\Delta t)$ = # of rabbits at time $t+\Delta t$.

How is $P(t+\Delta t)$ related to $P(t)$?

$P(t+\Delta t) = P(t)$ + "bunnies born between time t and time $t+\Delta t$"

− "rabbits who died between time t and time $t+\Delta t$"

Nothing profound there.

How can we guess at the # of bunnies born & the # of rabbits who died?

Malthus' approximation:

"the # of bunnies born is proportional to the number of rabbits who're there to breed. 10 rabbits will produce fewer bunnies than 100 rabbits will. And the larger the interval of time, Δt, the more bunnies there will be."

Mathematically, this means

the amount of time elapsed

$$\text{"bunnies born"} = k_B \, P(t) \, \Delta t$$

\# of rabbits present at time t

a number that reflects fertility, food availability, gestation period etc.

How do you approximate the # of rabbits who died between time t and Δt. Again, Malthus would argue that the # of rabbits who died is directly proportional to the # of rabbits who're around to die.

how long the period of time elapsed

$$\text{"rabbits who died"} = k_D \, P(t) \, \Delta t$$

\# of rabbits present at time t

a number that reflects health, food availability, # of predators

Putting this all together,

Malthus would suggest

$$P(t+\Delta t) = P(t) + k_B P(t)\Delta t - k_D P(t)\Delta t$$
$$= P(t) + (k_B - k_D) P(t)\Delta t$$

Note: if more bunnies are born than rabbits die, we'll have $k_B - k_D > 0$. This means that $P(t+\Delta t)$ will always be bigger than $P(t)$. As time passes, the population will grow. Similarly, if the death rate exceeds the birth rate (too many predators around, for example) then $k_B - k_D < 0$ and $P(t+\Delta t)$ will always be smaller than $P(t)$. As time passes, the population will decrease.

This is the same as

$$P(t+\Delta t) - P(t) = (k_B - k_D) P(t)\Delta t$$

$$\frac{P(t+\Delta t) - P(t)}{\Delta t} = (k_B - k_D) P(t)$$

If we want to model the population using a differential equation, then we take $\Delta t \to 0$.

$$\frac{P(t+\Delta t)-P(t)}{\Delta t} \longrightarrow \frac{dP}{dt} \qquad \text{as } \Delta t \to 0$$

and so we could model the rabbit population with

$$\boxed{\frac{dP}{dt} = (k_B - k_D)\,P}$$

The book writes $\frac{dP}{dt} = kP$, taking $k = k_B - k_D$.

Just remember that k reflects both births and deaths. Its <u>sign</u> is the important thing.

<u>Radioactive Decay</u>.

Here, just think of the # of uranium isotopes. They break down and so there's definitely a "death process". Can new isotopes be created? i.e. is there a "birth process"? Yes, there can be. But usually the problems you run across have to with the "death process" only:

$$\boxed{\frac{dP}{dt} = -k_D P}$$

There are various other systems in which one ends up with an equation of the form

$$\frac{dP}{dt} = kP \quad \text{or} \quad \frac{dP}{dt} = k(P - P_s)$$

a constant

Both are separable and both can be explicitly solved

$\frac{dP}{dt} = kP$ has general solution $P = Ce^{kt}$

$\frac{dP}{dt} = k(P-P_s)$ has general solution $P = P_s + Ce^{kt}$

The question is always: How do you find C? k? P_s?
You need all of them in order to be able to find P.

#4 A bacteria culture grows with constant relative growth rate. After 2 hours, there are 600 bacteria and after 8 hours there are 75,000.

a) find the initial population

b) find an expression for the population after t hours

c) find the population after 5 hours

d) find the rate of growth after 5 hours

e) when will the population reach 200,000?

translation: relative growth rate = k

rate of growth at time t_0 = $\frac{dP}{dt}$ at time t_0

We know that

$$P(t) = Ce^{kt}$$

we don't know C or k.

We're told $P(2) = Ce^{k2} = 600$

$P(8) = Ce^{k\cdot 8} = 75{,}000$

$$\frac{P(8)}{P(2)} = \frac{Ce^{8k}}{Ce^{2k}} = e^{6k} = \frac{75,000}{600} = \frac{750}{6} = 125$$

$$e^{6k} = 125 \Rightarrow 6k = \ln(125)$$

$$\Rightarrow k = \frac{\ln(125)}{6} \cong 0.805$$

$$= \frac{\ln(5^3)}{6} = \frac{\ln(5)}{2}$$

Great! We found k! Now to find C...

$$P(2) = 600 = Ce^{k2} = Ce^{2\left(\frac{\ln(5)}{2}\right)} = Ce^{\ln(5)}$$

$$\Rightarrow 600 = C \cdot 5 \Rightarrow \boxed{C = 120}$$

Done! $P = 120e^{t\frac{\ln(5)}{2}}$

ⓐ Initial population? take $t=0$.. $\boxed{P(0) = 120}$

ⓑ Population after t hours? $\boxed{P(t) = 120\,e^{t\frac{\ln(5)}{2}}}$

ⓒ Population after 5 hours? $\boxed{P(5) = 120\,e^{\frac{5\ln(5)}{2}}}$

d rate of growth after 5 hours?

$$\frac{dP}{dt}(5) = k\,P(5) = \frac{\ln(5)}{2} \cdot 120\, e^{\frac{5\ln(5)}{2}}$$

$$= 60 \ln 5\, e^{\frac{5\ln(5)}{2}}$$

e when will population equal 200,000?

$$P(t_0) = 200{,}000$$

$$\Rightarrow \quad 200{,}000 = 120\, e^{t_0 \frac{\ln(5)}{2}}$$

$$\Rightarrow \quad \frac{200{,}000}{120} = e^{t_0 \frac{\ln(5)}{2}}$$

$$\Rightarrow \quad \frac{5000}{3} = e^{t_0 \frac{\ln(5)}{2}}$$

$$\Rightarrow \quad \ln\left(\frac{5000}{3}\right) = t_0 \frac{\ln(5)}{2}$$

$$\Rightarrow \quad t_0 = \frac{2}{\ln(5)} \ln\left(\frac{5000}{3}\right) \cong 9.22 \text{ hours}$$

\# 16 A freshly brewed cup of coffee has temperature 95°. It's in a 20° room. When its temperature is 70°, it is cooling at a rate of 1° per minute. When does that occur?

$$\frac{dP}{dt} = k(P - P_s) \Rightarrow P = P_s + Ce^{kt}$$

We know that if you wait longer & longer the coffee temp will get closer & closer to room temperature (20°).

$$\lim_{t\to\infty} P = \lim_{t\to\infty}\left(P_s + Ce^{kt}\right) = \begin{cases} P_s & \text{if } k<0 \\ +\infty \text{ or } -\infty & \text{if } k>0 \end{cases}$$

So we hope that k will turn out to be negative and take $P_s = 20$.

We're given $\frac{dP}{dt}$ when $P = 70$. Specifically, when $P = 70$, $\frac{dP}{dt} = -1$. (Why the negative sign? "cooling" means $\frac{dP}{dt} < 0$!)

So when $P = 70$ $\frac{dP}{dt} = -1$

$$\Rightarrow \quad -1 = \frac{dP}{dt} = k(P - P_s) = k(70-20) = 50k$$

$$\Rightarrow \quad k = -1/50 \quad !$$

We're almost done!

$$P(t) = 20 + Ce^{(-1/50)t}$$

We just need to find C. We're told that initially, $P = 95$. So when $t = 0$, we know $P = 95$

$$\Rightarrow \quad 95 = P(0) = 20 + Ce^{-\frac{1}{50}t} = 20 + Ce^{-\frac{1}{50}\cdot 0}$$

$$= 20 + C$$

$$\Rightarrow \boxed{C = 75}$$

Hence $P = 20 + 75e^{-\frac{1}{50}t}$. Oof! We were asked how many minutes have passed when $P = 70$.

$$P(t_0) = 70 = 20 + 75e^{-\frac{1}{50}t_0} \Rightarrow 50 = 75e^{-\frac{1}{50}t_0}$$

$$\Rightarrow \frac{2}{3} = e^{-\frac{1}{50}t_0} \Rightarrow \ln\left(\frac{2}{3}\right) = -\frac{1}{50}t_0$$

$$\Rightarrow \boxed{-50 \ln\left(\frac{2}{3}\right) = t_0 \cong 20.3 \text{ minutes}}$$

#18 a) if \$500 is borrowed at 14% interest find the amounts due at the end of 2 years if the interest is compounded (i) annually (ii) quarterly (iii) monthly, (iv) daily, (v) hourly, (vi) continuously.

You borrow \$500. If the interest is compounded annually, then at the end of 1 year, you owe

$$\underbrace{500}_{\text{principal}} + \underbrace{(.14)500}_{\text{interest.}} = (1.14)500$$

after 2 years, you owe

$$[(1.14)500] + (.14)[(1.14)500]$$

$$= (1.14)\left[(1.14)\,500\right] = (1.14)^2\,500$$

why? after one year your principal is $(1.14)\,500$.
The interest on this is $.14\left[(1.14)\,500\right]$

(i) compounded annually $\Rightarrow (1.14)^2\,500 = 649.80$ \$

Compounded quarterly? If the annual interest rate is 14% then the quarterly interest rate is $\frac{14}{4}$ %.

owe after 1 quarter : $\left(1 + \frac{.14}{4}\right)500$

owe after 2 quarters : $\left(1+\frac{.14}{4}\right)\left[\left(1+\frac{.14}{4}\right)500\right]$

$$= \left(1+\frac{.14}{4}\right)^2 500$$

owe after N quarters : $\left(1+\frac{.14}{4}\right)^N 500$.

8 quarters in 2 years $\Rightarrow$ owe $\left(1+\frac{.14}{4}\right)^8 500 \cong 658.40$ \$

Compounded monthly? if the annual interest rate is 14% the monthly interest rate is $\frac{14}{12}\%$

owe after 1 month: $\left(1+\frac{.14}{12}\right) 500$

after 2 months: $\left(1+\frac{.14}{12}\right)^2 500$

after N months: $\left(1+\frac{.14}{12}\right)^N 500$

there are 24 months in 2 years so after 2 years, owe $\left(1+\frac{.14}{12}\right)^{24} 500 \cong 660.49\ \$$

Compounded daily? Annual interest rate = 14%
then daily interest rate is $\frac{14}{365.25}\%$ (approx.)

owe after N days: $\left(1+\frac{.14}{365.25}\right)^N 500$

there are 730.5 days in 2 years (approx.)

so after 2 years owe

$$\left(1+\frac{.14}{365.25}\right)^{730.5} 500 \cong 661.53\ \$$$

compounded hourly?

There are $(365.25)24 = 8766$ hours in a year. (approx.)

$\Rightarrow$ owe after N hours : $\left(1+\frac{.14}{8766}\right)^N 500$

owe after 2 years : $\left(1+\frac{.14}{8766}\right)^{17532} 500 \cong 661.57\$$

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compounded continuously? Divide the year up into n intervals. Then the interest rate is $\frac{14}{n}\%$

and after N of these intervals, you owe

$$\left(1+\frac{.14}{n}\right)^N 500$$

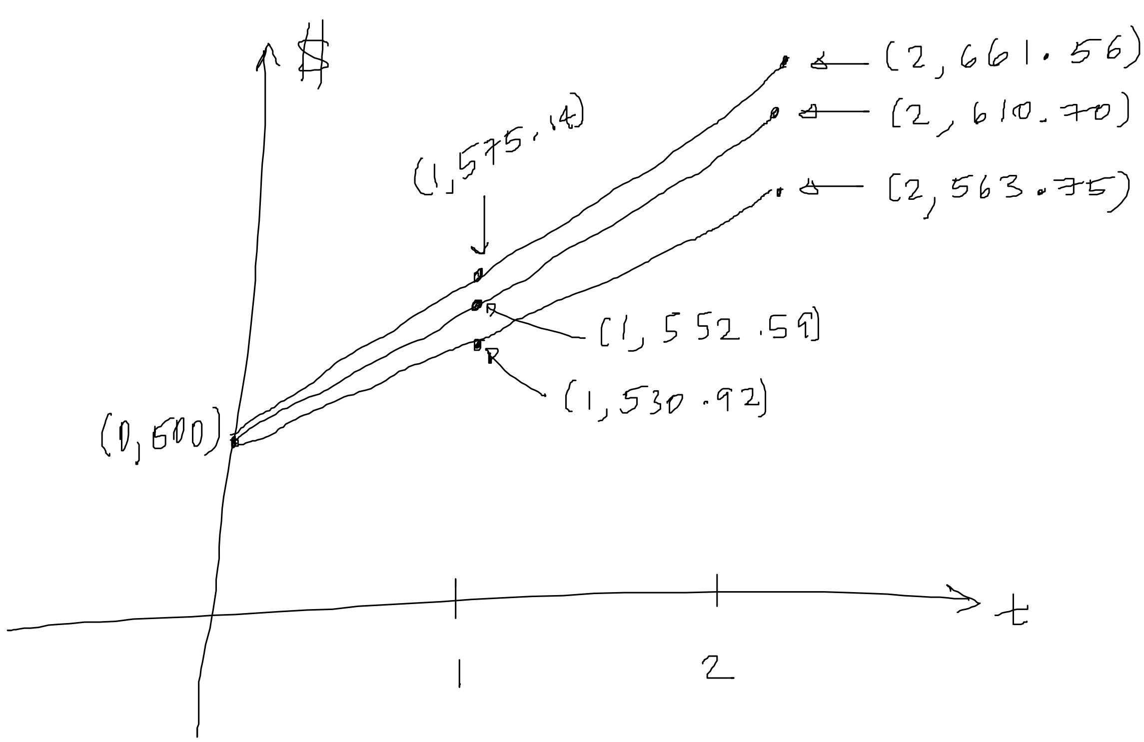
In 2 years, there are $2n$ intervals, so after 2 years you owe $\left(1+\frac{.14}{n}\right)^{2n} 500$. Now take $n \to \infty$

$$\lim_{n\to\infty}\left(1+\frac{.14}{n}\right)^{2n} 500 = 500\, e^{2(.14)} \cong 661.56\$$$

So now you see why banks compound interest daily!

18 b Suppose 500\$ is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years where $0 \le t \le 2$ graph $A(t)$ for the interest rates 14%, 10%, and 6%

i.e. graph $500e^{.14t}$, $500e^{.10t}$, $500e^{.06t}$



Interest rates matter!