Feb 21, 2005 § 9.3 Separable Equations

A <u>separable equation</u> is a first order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x times a function of y:

$$\frac{dy}{dx} = g(x)\, f(y)$$

<u>separable:</u>

$$\frac{dy}{dx} = 3x \qquad \frac{dy}{dx} = 2y+1$$

$$\frac{dy}{dx} = \cos(x)\,(y^2+1) \qquad \frac{dy}{dx} = y^2 + xy^2$$

$$\frac{dy}{dx} = ky\left(1 - \frac{y}{K}\right) \longleftarrow \text{the logistic equation}$$

<u>not separable:</u>

$$\frac{dy}{dx} = y^2 + xy$$

$$\frac{dy}{dx} = \cos(x+y)$$

The good news is that if an equation is separable then you may be able to find the general solution using methods from integration.

Here's how:

$$\frac{dy}{dx} = g(x) f(y)$$

$$\Rightarrow \quad \frac{dy}{f(y)} = g(x)\,dx$$

$$\Rightarrow \quad \int \frac{1}{f(y)}\,dy = \int g(x)\,dx$$

if we can find the antiderivatives $\frac{1}{f(y)}$ and $g(x)$ then this will help us find the general solution of $\frac{dy}{dx} = g(x) f(x)$

ex: $\frac{dy}{dx} = 3x$ we already know the solution! $y(x) = \frac{3}{2}x^2 + C$

Using the general method for separable equations,

$$dy = 3x\,dx$$

$$\Rightarrow \int dy = \int 3x\,dx$$

$$\Rightarrow y + C = \frac{3}{2}x^2 + D$$

$$\Rightarrow y = \frac{3}{2}x^2 + D - C$$

$$\Rightarrow y = \frac{3}{2}x^2 + C$$

since the right hand side really would have $D-C$ but $D-C$ is just a constant, so I've renamed it C.

ex: $\frac{dy}{dx} = ky$ we already know the general solution. It's $y(x) = Ce^{kx}$

$$\frac{dy}{y} = k\,dx \Rightarrow \int \frac{dy}{y} = \int k\,dx$$

$$\Rightarrow \ln(y) + C = kx + D$$

$$\Rightarrow \ln(y) = kx + C$$ (same logic as before w/ the constants)

$$\Rightarrow y = e^{kx + C}$$

$$= e^{C} e^{kx}$$

$$\Rightarrow y = Ce^{kx}$$ (same logic as before w/ the constant)

ii:

$$\frac{dy}{dx} = ky\left(1 - \frac{y}{K_1}\right)$$

from before, $y(x) = \dfrac{Ce^{kx}K_1}{K_1 + Ce^{kx}}$

$$\frac{dy}{ky\left(1 - \frac{y}{K_1}\right)} = dx$$

$$\Rightarrow \int \frac{dy}{ky\left(1 - \frac{y}{K_1}\right)} = \int dx$$

$$\Rightarrow \int \frac{1}{ky} + \frac{1}{K_1 k\left(1 - y/K_1\right)}\, dy = \int dx$$

$$\Rightarrow \frac{\ln|y|}{k} - \frac{\ln\left(1 - y/K_1\right)}{k} = x + C_1$$

$$\Rightarrow \ln\left(\frac{y}{1 - y/K_1}\right) = kx + C_1$$

$$\Rightarrow \quad \frac{y}{1 - y/K_1} = Ce^{kx}$$

(note that I'm replacing the constant e^C with C.)

now this is something I can solve for in terms of y

$$y = \frac{Ce^{kx} K_1}{K_1 + Ce^{kx}}$$

as before!

ex:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \quad y\,dy = -x\,dx$$

$$\Rightarrow \int y\,dy = \int -x\,dx$$

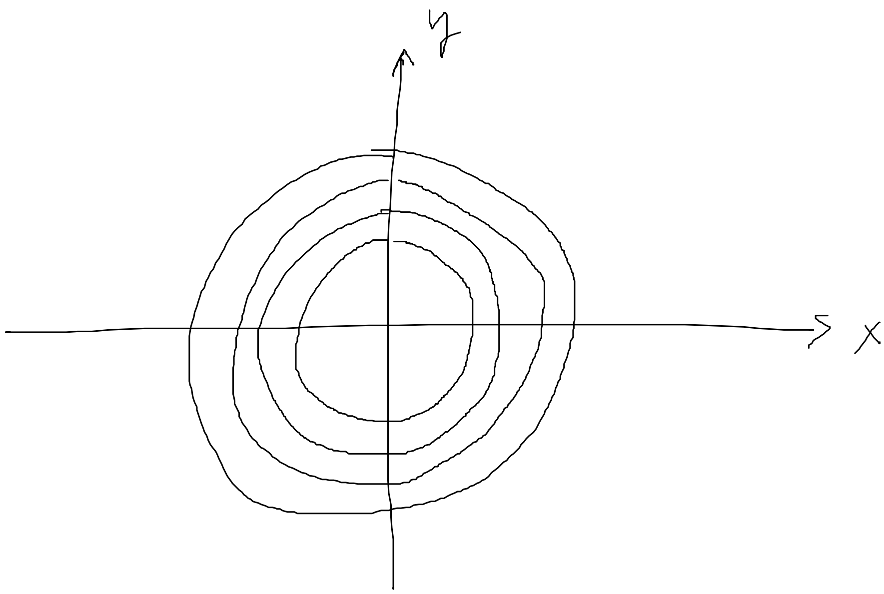
$$\Rightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\Rightarrow \quad y^2 = -x^2 + r^2$$

$$\Rightarrow \quad x^2 + y^2 = r^2$$

circles!

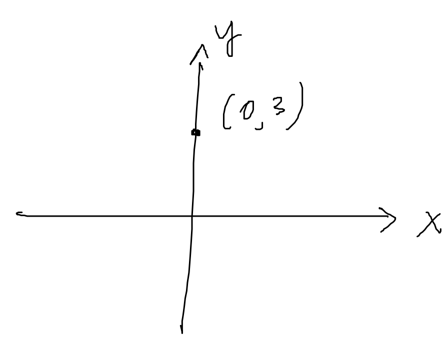
we know that the general solutions will have to be on circles.



How do we find y?

If the initial value problem is

$y(0) = 3$



$y^2 + x^2 = 9$ ← Why 9? Because we've been told $y = 3$ when $x = 0$

we know that y is positive for x near 0

so $y = \sqrt{9 - x^2}$

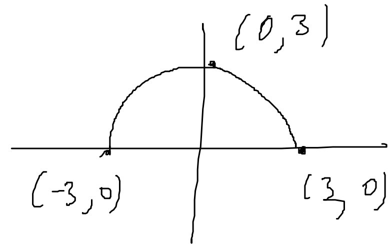
this means that the solution of

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{with initial data } y(0) = 3$$

is

$$y(x) = \sqrt{9 - x^2}$$

plotting it:

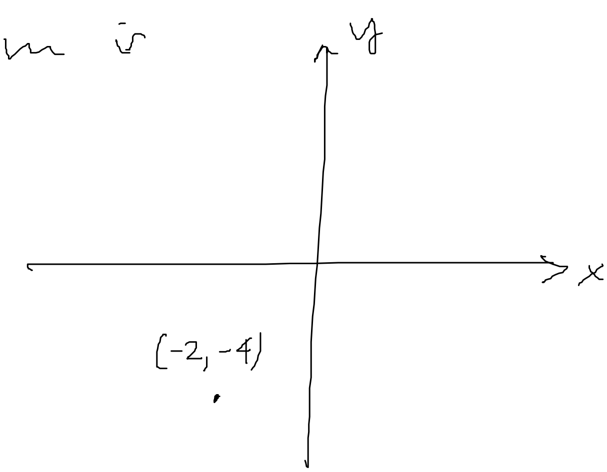


note that it only exists for $-3 \le x \le 3$ and satisfies the differential equation for $-3 < x < 3$.

if the initial value problem is

$$y(-2) = -4$$

then the solution is on the circle



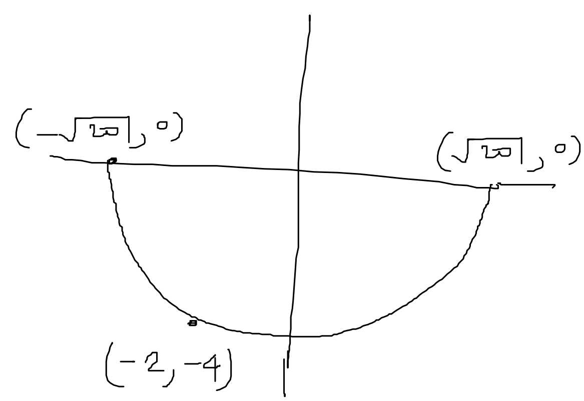
$$y^2 + x^2 = 20$$

why? since we're told that when $x = -2$, $y = -4$. That is,

$$(-4)^2 + (-2)^2 = r^2 \Rightarrow r^2 = 20$$

since y is negative at $x = -2$, y will be negative near $x = -2$. The solution of the initial value problem is

$$y(x) = -\sqrt{20 - x^2}$$



note that

the solution only exists for

$$-\sqrt{20} \leq x \leq \sqrt{20}$$

and satisfies the differential equation for $-\sqrt{20} < x < \sqrt{20}$.

In general, the solution of

$$\frac{dy}{dx} = -\frac{x}{y} \quad \text{with initial data } y(x_0) = y_0$$

is

$$y = \begin{cases} +\sqrt{x_0^2 + y_0^2 - x^2} & \text{if } y_0 > 0 \\ -\sqrt{x_0^2 + y_0^2 - x^2} & \text{if } y_0 < 0 \end{cases}$$

its graph is either the upper half or the lower half of a circle of radius $x_0^2 + y_0^2$

ex: $$\frac{dy}{dx} = -\frac{(x-a)}{(y-b)}$$

as before, we find the solution by the method for separable equations.

$$(y-b)\,dy = -(x-a)\,dx$$

$$\Rightarrow\Rightarrow\Rightarrow \quad (x-a)^2 + (y-b)^2 = r^2$$

we'll have the same types of solutions as described above