Mat 135 Jan 31, 2005

§7.4 Integration of Rational Functions by Partial Fractions

Rational function: any ratio of polynomials.

e.g. $\dfrac{2x+1}{x^2+4}$ is a rational function

$$\int \frac{2x+1}{x^2+4}\,dx = \int \frac{2x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= \int \frac{1}{u}\,du + \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$u = x^2+4$

$$= \ln(x^2+4) + \frac{1}{2}\int \frac{1}{u^2+1}\,du$$

$u = x/2$

$$= \boxed{\ln(x^2+4) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

In general, can we compute

$$\int \frac{p(x)}{Q(x)}\, dx$$

if $P(x)$ and $Q(x)$ are polynomials? Yes!

step 1: do long division so that

$$\frac{p(x)}{Q(x)} = P_0(x) + \frac{P_1(x)}{Q(x)}$$

where degree $P_1(x)$ < degree $Q(x)$.

e.g. $$\frac{3x^3+x^2-x}{x^2-1} = 3x+1 + \frac{2x+1}{x^2-1}$$

$$\frac{4x^5-9x^4+9x^3-14x^2+6x-5}{x^3-2x^2+x-2} = 4x^2-x+3+\frac{x^2+x+1}{x^3-2x^2+x-2}$$

step 2: factor the denominator into linear and irreducible quadratic factors.

e.g.

$$3x+1+\frac{2x+1}{(x^2-1}=3x+1+\frac{2x+1}{(x+1)(x-1)}$$

$$4x^2-x+3+\frac{x^2+x+1}{x^3-2x^2+x-2}=4x^2-x+3+\frac{x^2+x+1}{(x-2)(x^2+1)}$$

Q: What's an irreducible quadratic factor?

A: A quadratic polynomial that has no real roots. For example

$$x^2+1,\quad x^2-4x+5,\quad x^2-6x+13$$

Step 3: Using the form of the denominator, write the rational function as a sum of rational functions you know how to integrate.

Huh? What's that mean?

$$3x+1+\frac{2x+1}{(x+1)(x-1)} = 3x+1+\frac{A}{x+1}+\frac{B}{x-1}$$

find A & B ...

$$\frac{2x+1}{(x+1)(x-1)} = \frac{A}{x+1}+\frac{B}{x-1} = \frac{A(x-1)+B(x+1)}{(x+1)(x-1)}$$

$$= \frac{Ax - A + Bx + B}{(x+1)(x-1)}$$

so we need $2x = Ax + Bx \Rightarrow 2 = A+B$

$1 = -A + B$

find A and B so that $\begin{cases} A+B=2 \\ -A+B=1 \end{cases} \Rightarrow A = 1/2,\ B = 3/2$

thus $\displaystyle\int \frac{3x^3+x^2-x}{x^2-1}\,dx = \int 3x+1+\frac{2x+1}{x^2-1}\,dx$

$$= \frac{3}{2}x^2 + x + \int \frac{1/2}{x+1} + \frac{3/2}{x-1}\,dx$$

$$= \boxed{\frac{3}{2}x^2 + x + \frac{1}{2}\ln(x+1) + \frac{3}{2}\ln(x-1) + C}$$

What about the other case?

$$4x^2 - x + 3 + \frac{x^2 + x + 1}{(x-2)(x^2+1)}$$

Here, you'll want to find A, B, C so that

$$\frac{A}{x-2} + \frac{Bx + C}{x^2+1} = \frac{x^2 + x + 1}{(x-2)(x^2+1)}$$

that is,

$$\frac{A(x^2+1) + (Bx+c)(x - \cancel{B}2)}{(x-2)(x^2+1)} = \frac{x^2+x+1}{(x-2)(x^2+1)}$$

$$\frac{Ax^2 + A + Bx^2 - 2Bx + Cx - 2C}{(x-2)(x^2+1)} = \frac{x^2 + x + 1}{(x-2)(x^2+1)}$$

So $\quad Ax^2 + Bx^2 = x^2 \quad \Rightarrow \quad A + B = 1$

$\quad -2Bx + Cx = x \quad \Rightarrow \quad -2B + C = 1$

$\quad A - 2C = 1$

seek A, B, C so that $\begin{cases} A + B = 1 \\ -2B + C = 1 \\ A - 2C = 1 \end{cases} \Rightarrow \begin{array}{l} A = 7/5 \\ B = -2/5 \\ C = 1/5 \end{array}$

So

$$\int \frac{4x^5-9x^4+9x^3-14x^2+6x-5}{x^3-2x^2+x-2}\,dx$$

$$= \int 4x^2 - x + 3 + \frac{x^2+x+1}{(x-2)(x^2+1)}\,dx$$

$$= \frac{4}{3}x^3 - \frac{1}{2}x^2 + 3x + \int \frac{7/5}{x-2} + \frac{-2/5\,x + 1/5}{x^2+1}\,dx$$

$$= \frac{4}{3}x^3 - \frac{1}{2}x^2 + 3x + \frac{7}{5}\ln(x-2) - \frac{1}{5}\int \frac{2x}{x^2+1}\,dx$$

$$+ \frac{1}{5}\int \frac{1}{x^2+1}\,dx$$

$$= \boxed{\frac{4}{3}x^3 - \frac{1}{2}x^2 + 3x + \frac{7}{5}\ln|x-2| - \frac{1}{5}\ln(x^2+1) + \frac{1}{5}\tan^{-1}(x) + C}$$

<u>Note!</u> we needed 3 free parameters A, B, C to match the 3 coefficients of the numerator

$$1\cdot x^2 + 1\cdot x + 1\cdot 1$$

match these

see the book for what to do when the factors repeat:

eg $$\frac{x^3 - x + 1}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

$$\frac{x^4 + x - 2}{x(2x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}$$

One more example:

$$\frac{3x^5 - 25x^4 + 84x^3 - 110x^2 + 49x - 25}{x^3 - 8x^2 + 25x - 26}$$

$$= 3x^2 - x + 1 + \frac{x^2 - 2x + 1}{x^3 - 8x^2 + 25x - 26}$$

$$= 3x^2 - x + 1 + \frac{x^2 - 2x + 1}{(x-2)(x^2 - 6x + 13)}$$

find A, B, C such that

$$\frac{A}{x-2} + \frac{Bx+C}{x^2 - 6x + 13} = \frac{x^2 - 2x + 1}{(x-2)(x^2 - 6x + 13)}$$

that is

$$\frac{A(x^2-6x+13)+(Bx+C)(x-2)}{(x-2)(x^2-6x+13)} = \frac{x^2-2x+1}{(x-2)(x^2-6x+13)}$$

for this, we need

$$Ax^2-6Ax+13A+Bx^2-2Bx+Cx-2C = x^2-2+1$$

equating coeffiencents, we need

$$A+B=1$$
$$-6A-2B+C=-2$$
$$13A-2C=1$$

solving for A, B, C find

$$A = 1/5 \quad B = \frac{4}{5} \quad C = 4/5$$

So

$$\int \frac{x^2-2x+1}{(x-2)(x^2-6x+13)}\,dx = \int \frac{1/5}{x-2} + \frac{4/5\,x + 4/5}{x^2-6x+13}\,dx$$

$$= \frac{1}{5}\ln(x-2) + \frac{4}{5}\int \frac{x+1}{x^2-6x+13}\,dx$$

Now what ?!?

$$\int \frac{x+1}{x^2-6x+13}\,dx = \int \frac{x-3+3+1}{x^2-6x+13}\,dx$$

$$= \int \frac{x-3}{x^2-6x+13}\,dx + 4\int \frac{1}{x^2-6x+13}\,dx$$

$$u = x^2-6x+13$$
$$du = 2x-6$$

$$= \frac{1}{2}\ln(x^2-6x+13) + 4\int \frac{1}{x^2-6x+13}\,dx$$

$$= \frac{1}{2}\ln(x^2-6x+13) + 4\int \frac{1}{(x-3)^2+4}\,dx$$

$$= \frac{1}{2}\ln(x^2-6x+13) + \frac{4}{4}\int \frac{1}{\left(\frac{x-3}{2}\right)^2+1}\,dx$$

$$u = \frac{x-3}{2}$$

$$= \frac{1}{2}\ln(x^2-6x+13) + 2\int \frac{du}{u^2+1} \quad \text{where } u = \frac{x-3}{2}$$

$$= \frac{1}{2}\ln(x^2-6x+13) + 2\tan^{-1}\left(\frac{x-3}{2}\right) + C$$

Putting it all together,

$$\int \frac{3x^5 - 25x^4 + 84x^3 - 110x^2 + 49x - 25}{x^3 - 8x^2 + 25x - 26}$$

$$= x^3 - \frac{1}{2}x^2 + x + \frac{1}{5}\ln(x-2)$$

$$+ \frac{4}{10}\ln(x^2 - 6x + 13)$$

$$+ \frac{8}{5}\tan^{-1}\left(\frac{x-3}{2}\right) + C$$

Q: How did you find

$$x^2 - 6x + 13 = (x-3)^2 + 4 \quad ??$$

A: the roots will be complex, of the form $a + ib$ and $a - ib$

so $(x^2 - 6x + 13) = (x - (a+ib))(x - (a-ib))$

$$= ((x-a) - ib)((x-a) + ib)$$

$$= (x-a)^2 + b^2$$

by the quadratic formula, the roots are $3 \pm 2i$

And so,

$$x^2 - 6x + 13 = (x-3)^2 + 2^2$$

$$= (x-3)^2 + 4$$

as claimed