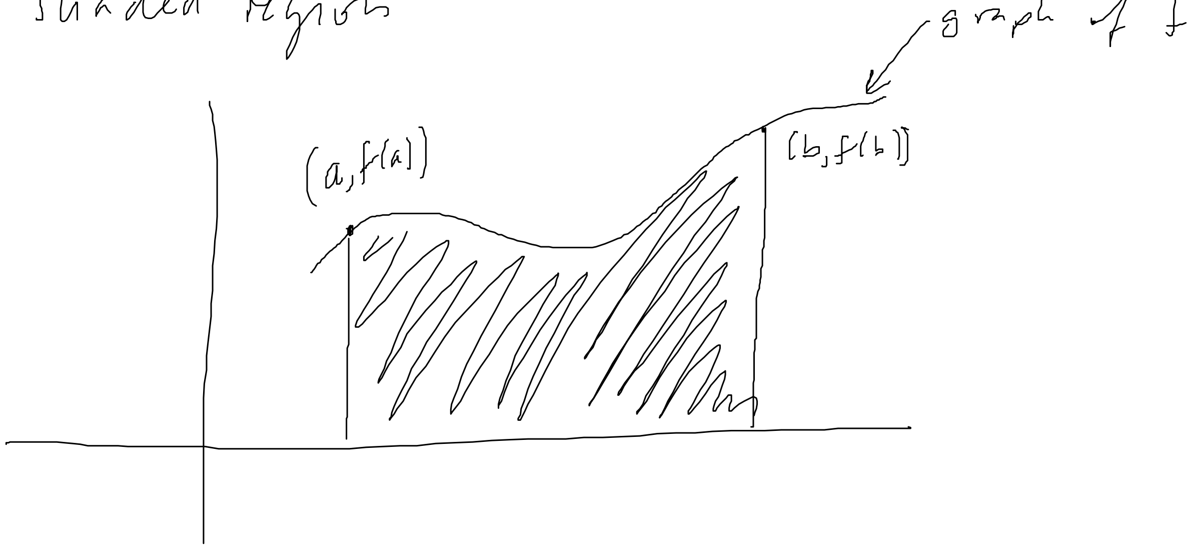
MAT 135 Dec 3, 2004

§5.1 Continued.

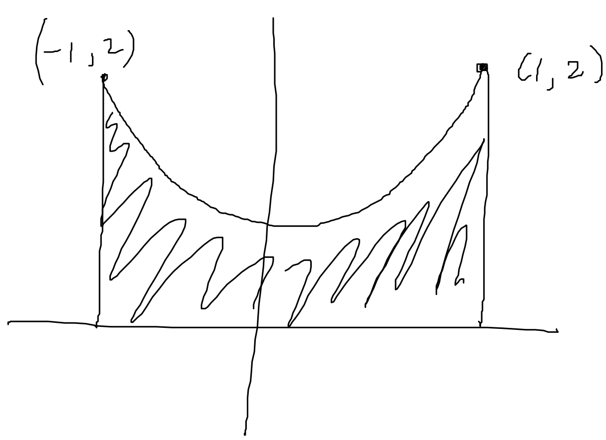
If you want to find the area of the shaded region



then how can you do this?

A: Approximate the region with rectangles. Compute the total area of the rectangles. And then, take the # of rectangles to infinity

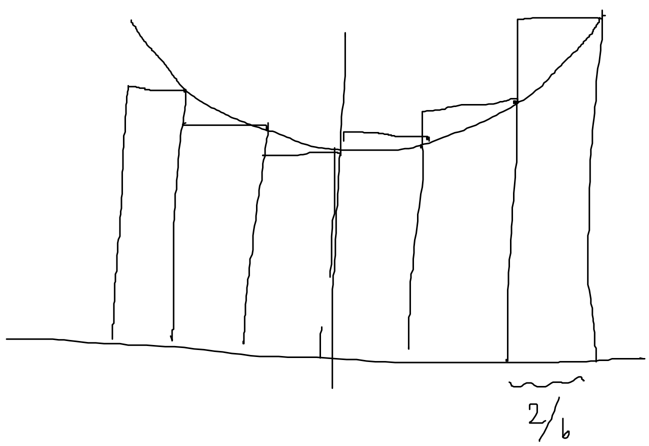
Consider $f(x) = x^2+1$ on $[-1, 1]$



There are lots of ways to approximate with rectangles.

Consider doing it with 6 rectangles of equal width.

$$L_6 = \frac{2}{6}\cdot f(-1) + \frac{2}{6} f\left(-\frac{2}{3}\right) + \frac{2}{6} f\left(-\frac{1}{3}\right) + \frac{2}{6} f(0) + \frac{2}{6} f\left(\frac{1}{3}\right) + \frac{2}{6} f\left(\frac{2}{3}\right)$$



$$R_6 = \frac{2}{6} f\left(-\frac{2}{3}\right) + \frac{2}{6} f\left(-\frac{1}{3}\right) + \frac{2}{6} f(0) + \frac{2}{6} f\left(\frac{1}{3}\right) + \frac{2}{6} f\left(\frac{2}{3}\right) + \frac{2}{6} f(1)$$

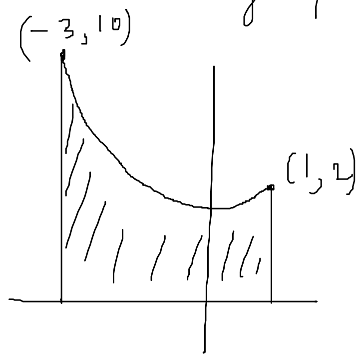
In this case,

$$L_6 = \frac{73}{27}$$

$$R_6 = \frac{73}{27}$$

they are equal because x^2+1 is an even function on $[-1, 1]$

usually that won't happen. For example, if you were approximating the area



$f(x) = x^2+1$

then $L_6 = 440/27$

$R_6 = 296/27$

The hope is that if you take more and more rectangles then

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n = \text{area under the curve.}$$

Let's continue w/ the example of

$f(x) = x^2+1$ on $[-3, 1]$

n	L_n	R_n
2	24	8
4	18	10
8	$\frac{31}{2} = 15.5$	$23/2 = 11.5$
16	$115/8 = 14.375$	$99/8 = 12.375$
32	$443/32 \cong 13.8438$	$411/32 \cong 12.8438$
64	$1739/128 \cong 13.5859$	$1675/128 \cong 13.0859$
128	$6891/512 \cong 13.4590$	$6763/512 \cong 13.2090$
256	$27435/2048 \cong 13.3960$	$27179/2048 \cong 13.2710$
512	$109483/8192 \cong 13.3646$	$108971/8192 \cong 13.3021$
1024	$437419/32768 \cong 13.3490$	$436395/32768 \cong 13.3177$

there is a limit

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} R_n$$

$$= \frac{40}{3} \cong 13.3333$$

To continue, we need to introduce notation.

First, if you want to approximate

with n rectangles of equal width, how do you describe these rectangles?

first: width $= \Delta x := \dfrac{b-a}{n}$

then divide $[a,b]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots [x_{n-1}, x_n]$$

where
$$x_0 = a$$
$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$x_3 = a + 3\Delta x$$
$$\vdots$$
$$x_i = a + i\Delta x$$
$$\vdots$$
$$x_n = a + n\Delta x = a + n\left(\frac{b-a}{n}\right) \overset{\checkmark}{=} b$$

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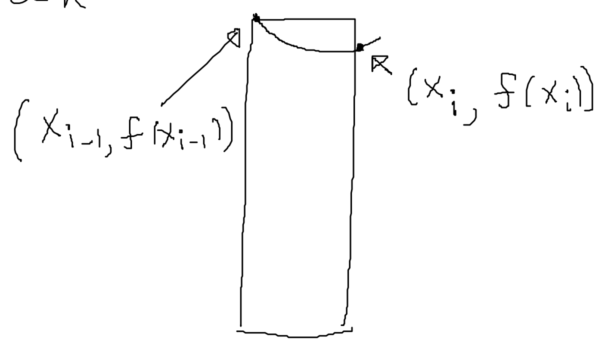
If you're doing the left approximation, you need



The area of the i^{th} rectangle is

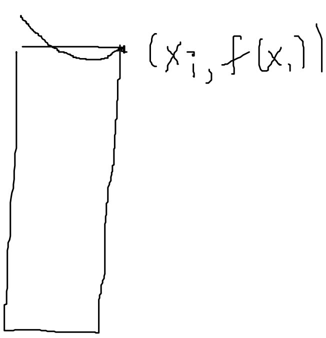
height × base $= \Delta x \; f(x_{i-1})$

and so

$$L_n = \begin{pmatrix}\text{area } 1^{st} \\ \text{rect.}\end{pmatrix} + \begin{pmatrix}\text{area } 2^{nd} \\ \text{rect.}\end{pmatrix} + \begin{pmatrix}\text{area } 3^{rd} \\ \text{rect.}\end{pmatrix} + \dots + \begin{pmatrix}\text{area } n^{th} \\ \text{rect}\end{pmatrix}$$

$$= f(x_0)\,\Delta x + f(x_1)\,\Delta x + f(x_2)\,\Delta x + \dots + f(x_{n-1})\,\Delta x$$

If you're doing the right-endpoint approximation then the area of the i^{th} rectangle is



$f(x_i)\Delta x$

and so

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \dots + f(x_n)\Delta x$$

We introduce "sigma notation"

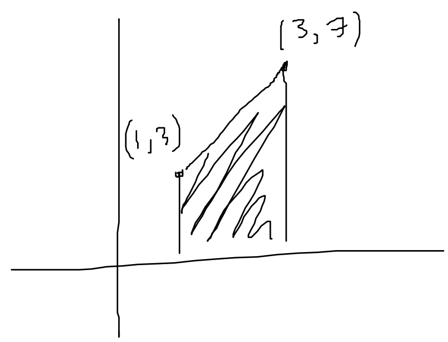
$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x$$

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

and hope that $\lim_{n\to\infty} L_n$ exists, $\lim_{n\to\infty} R_n$ exists

and that these limits are equal. If this happens then the area is the value of the limit.

example: $f(x) = 2x+1$ on $[1,3]$



you know the answer by geometry

area $= \square + \triangle = 2\times 3 + \frac{1}{2}(2)(4)$

$= 6 + 4 = 10$

$$\Delta x = \frac{3-1}{n} = \frac{2}{n}$$

$$x_i = 1 + i\Delta x$$

$$L_n = \sum_{i=0}^{n-1} f(x_i)\Delta x = \sum_{i=0}^{n-1} f(1+i\Delta x)\,\Delta x$$

$$= \sum_{i=0}^{n-1} \left[2(1+i\Delta x)+1\right]\Delta x$$

$$= \sum_{i=0}^{n-1} \left[2 + i2\Delta x + 1\right]\Delta x$$

$$= \sum_{i=0}^{n-1} \left[3\Delta x + i\,2(\Delta x)^2\right]$$

$$= \sum_{i=0}^{n-1} 3\Delta x + \sum_{i=0}^{n-1} i\,2(\Delta x)^2$$

$$= 3\Delta x \sum_{i=0}^{n-1} 1 + 2(\Delta x)^2 \sum_{i=0}^{n-1} i$$

$$\sum_{j=1}^{k} 1 = \underbrace{1+1+\dots+1}_{k \text{ times}} = k \qquad \sum_{j=1}^{k} j = 1+2+3+\dots+k = \frac{k(k+1)}{2}$$

and so

$$L_n = 3\Delta x\,(n) + 2(\Delta x)^2\,\frac{(n-1)(n)}{2}$$

and so

$$L_n = 3\Delta x\, n + (\Delta x)^2 (n-1)(n)$$

$$= 3\left(\frac{2}{n}\right)n + \left(\frac{2}{n}\right)^2 (n-1)(n)$$

$$= 6 + \frac{4(n-1)}{n}$$

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty} 6 + \frac{4(n-1)}{n} = 10$$

Similarly,

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x$$

$$= \sum_{i=1}^{n} \left[3\Delta x + i\,2(\Delta x)^2\right]$$

$$= 3\Delta x \sum_{i=1}^{n} 1 + 2(\Delta x)^2 \sum_{i=1}^{n} i$$

$$= 3\Delta x\, n + 2(\Delta x)^2 \frac{n(n+1)}{2}$$

$$= 6 + \frac{4(n+1)}{n}$$

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} 6 + \frac{4(n+1)}{n} = 10$$ ✓