Mat1062: Computational Methods for PDE

Mary Pugh

March 18, 2008

1 Convergence of the Approximate Solutions

We recall the weak problem: We have the Hilbert space

$$V = \{v(x) \text{ real-valued functions on } \mathbb{R} \;\Big|\; \int_0^1 v(x)^2\,dx < \infty, \int_0^1 v'(x)^2\,dx < \infty, v(0) = 0\Big\} \tag{1}$$

and if $f \in L^2([0,1])$ we seek $u \in V$ such that $a(u,v) = \langle f, v\rangle$ for all $v \in V$ where

$$a(u,v) = \int_0^1 u'(x)\,v'(x)\,dx \qquad \langle f, v\rangle = \int_0^1 f(x)\,v(x)\,dx$$

Given an n-dimensional subspace $V_n \subset V$ the Ritz-Galerkin approximation problem is: if $f \in L^2([0,1])$ we seek $u_n \in V_n$ such that $a(u_n, v) = \langle f, v\rangle$ for all $v \in V_n$. From last time, we saw how to use a basis of V_n to construct the approximate solution u_n.

We now turn to the question: does u_n converge to a weak solution u as $n \to \infty$?

1.1 Projections

We would like to know whether u_n is the projection of u onto the subspace V_n. That is, of all the elements of V_n is u_n the one that's closest to u? The answer turns out to be "yes" if we use the right inner product/norm. But before we prove this, we need some preliminaries.

We have two inner products on V: $a(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$. These inner products induce two norms:

$$\|u\| := \sqrt{\int_0^1 u^2} = \sqrt{\langle u, u\rangle} \qquad \|u\|_E := \sqrt{\int_0^1 u'^2} = \sqrt{a(u,u)}$$

We know $a(u_n, v) = \langle f, v \rangle$ for all $v \in V_n$. Similarly, we know $a(u, v) = \langle f, v \rangle$ for all $v \in V$. Hence $a(u, v) = \langle f, v \rangle$ for all $v \in V_n$. Therefore

$$a(u_n, v) = \langle f, v \rangle = a(u, v) \quad \forall\, v \in V_n \quad \Longrightarrow \quad a(u - u_n, v) = 0 \quad \forall\, v \in V_n$$

This shows that $u - u_n$ is perpendicular to the subspace V_n with respect to the $a(\cdot, \cdot)$ inner product. Given this observation, the following theorem is very natural.

Theorem *Assume $f \in L^2([0, 1])$, V_n is a subspace of V, u_n is a solution of the Ritz-Galerkin approximation problem and u is a weak solution. Then*

$$\|u - u_n\|_E = \inf_{v \in V_n} \|u - v\|_E.$$

Proof First of all, because $u_n \in V_n$ it's automatically true that

$$\inf_{v \in V_n} \|u - v\|_E \leq \|u - u_n\|_E.$$

We now prove the opposite inequality, which will then imply that the two sides are equal.

$$\begin{aligned} \|u - u_n\|_E^2 &= a(u - u_n, u - u_n) = a(u - u_n, u - v) + a(u - u_n, v - u_n) \\ &= a(u - u_n, u - v) \quad \forall\, v \in V_n. \end{aligned}$$

In the last step, we used that $v - u_n \in V_n$ and $u - u_n$ is perpendicular to V_n with respect to the $a(\cdot, \cdot)$ inner product. Applying the Scwartz inequality,

$$\|u - u_n\|_E^2 = a(u - u_n, u - v) \leq \|u - u_n\|_E \, \|u - v\|_E \quad \forall\, v \in V_n$$

Hence

$$\|u - u_n\|_E \leq \|u - v\|_E \quad \forall\, v \in V_n$$

Therefore

$$\|u - u_n\|_E \leq \inf_{v \in V_n} \|u - v\|_E,$$

finishing the proof.

Now we would like to bound $\|u - u_n\|_E$ with something that doesn't depend on u. We can do this under certain circumstances. For example, assume that our vector spaces satisfy:

Approximation Assumption *There is a constant ϵ such that for any $w \in V_n$*

$$\int_0^1 w_{xx}^2(x)\,dx < \infty \qquad \Longrightarrow \qquad \inf_{v\in V_n} \|w-v\|_E \le \epsilon \|w''\| \tag{2}$$

Not all elements of V_n will have $\int v_{xx}^2 < \infty$. The approximation assumption only applies to those vectors that have a second derivative and whose second derivative is square-integrable. If V and V_n are such that this approximation assumption holds then we can bound $\|u-u_n\|_E$ with something that doesn't depend on u. Specifically, we can prove an upper bound that depends on the data f:

Theorem *If $f \in L^2([0,1])$ and u is a weak solution and u_n is a solution of the Ritz-Galerkin approximation problem and if the approximation assumption (2) holds then*

$$\|u-u_n\| \le \epsilon \|u-u_n\|_E \le \epsilon^2 \|f\| \tag{3}$$

Proof: We use a duality argument to prove the first inequality. Let w be a weak solution of

$$\begin{cases} -w'' = u - u_n & \text{on } (0,1) \\ w(0) = 0 \\ w'(1) = 0 \end{cases}$$

Then

$$\begin{aligned} \|u-u_n\|^2 &= \int_0^1 (u(x)-u_n(x))(u(x)-u_n(x))\,dx \\ &= -\int_0^1 (u(x)-u_n(x))w''(x)\,dx = \int_0^1 (u'(x)-u_n'(x))w'(x)\,dx \\ &= a(u-u_n,w) = a(u-u_n, w-v) \quad \forall\, v \in V_n \end{aligned}$$

Therefore

$$\|u-u_n\|^2 \le \|u-u_n\|_E \|w-v\|_E \qquad \forall\, v \in V_n$$

hence

$$\|u-u_n\|^2 \le \|u-u_n\|_E \inf_{v\in V_n} \|w-v\|_E$$

The approximation assumption (2) applies (because $u - u_n \in L^2$ implies $w'' \in L^2$) and therefore

$$\|u - u_n\|^2 \leq \epsilon \|u - u_n\|_E \|w''\| = \epsilon \|u - u_n\|_E \|u - u_n\|$$

Dividing both sides of the inequality by $\|u - u_n\|$ yields the first inequality in (3). It remains to show that

$$\|u - u_n\|_E \leq \epsilon \|f\|.$$

We know by the prior theorem that

$$\|u - u_n\|_E = \inf_{v \in V_n} \|u - v\|_E$$

Because $f \in L^2$ we know that $u'' \in L^2$ and the approximation assumption (2) applies yielding

$$\inf_{v \in V_n} \|u - v\|_E \leq \epsilon \|u''\| = \epsilon \|f\|$$

proving the desired second inequality. This finishes the proof.

In the above proof, I was a little quick on a few details. For example, just because $f \in L^2$ it's not obvious that $u'' \in L^2$ unless we happen to know that u is actually a classical solution. If we knew that u were a classical solution then we'd know that $u'' = f$ and therefore $u \in L^2$. In fact, this is actually kosher for many elliptic PDE — a common approach for many elliptic PDE problems is: 1) Existence: prove that there is a weak solution using functional analysis methods, 2) Regularity: prove that the weak solution has enough derivatives to be a classical solution. I'm not going to go into this because this isn't a course in PDE.

1.2 Interpolants

Now that we know

$$\|u - u_n\| \leq \epsilon^2 \|f\|$$

for spaces that satisfy the approximation assumption (2) we'd love to know some spaces that do satisfy the assumption and, better yet, we'd like to know that $\epsilon \to 0$ as $n \to \infty$.

We recall the n-dimensional space of piece-wise linear functions. Fix a set of $n+1$ points in $[0,1]$ such that

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1.$$

We call these points "nodes". Using these $n+1$ nodes, we create n piecewise linear functions ϕ_j such that $\phi_j(x_j) = 1$ for all $1 \leq j \leq n$ and such that the support of ϕ_j is $[x_{j-1}, x_{j+1}]$.

Definition *Given v continuous on $[0,1]$, the interpolant of v is in V_n and*

$$v_I(x) := \sum_{j=1}^{n} v(x_j)\ \phi_j(x)$$

In other words, given v and a collection of nodes, v_I is what you get if you "connect-the-dots" on the graph using the node points $(x_j, v(x_j))$.

We start by proving what we observed in the computation: u_n and u agree at the nodes. To show this, it suffices to show that $u_n = u_I$.

Theorem: *If $f \in L^2([0,1])$ and u is a weak solution and u_n is a solution of the Ritz-Galerkin approximation problem and u_I is the interpolant of u in V_n then $u_n = u_I$.*

Proof: We prove this by showing that

$$a(u_I - u_n, v_n) = 0, \qquad \forall\ v_n \in V_n \tag{4}$$

Because $u_I - u_n \in V_n$ this will then imply $a(u_I - u_n, u_I - u_n) = 0$ which then implies $u_I - u_n = 0$, as desired.

To show (4) we prove

$$a(u - u_n, v_n) = a(u_I - u_n, v_n) \qquad \forall\ v_n \in V_n.$$

This would suffice because the left-hand side equals zero. We want to show

$$\int_0^1 (u - u_n)'\ v_n'\ dx = \int_0^1 (u_I - u_n)'\ v_n'\ dx \qquad \forall\ v_n \in V_n.$$

It suffices to show that for $1 \leq i \leq n$

$$\int_{x_{i-1}}^{x_i} (u - u_n)'\ v_n'\ dx = \int_{x_{i-1}}^{x_i} (u_I - u_n)'\ v_n'\ dx \qquad \forall\ v_n \in V_n.$$

Fix $v_n \in V_n$ and fix i. We know that v_n is piecewise linear on $[0,1]$ and is linear on $[x_{i-1}, x_i]$. (The corners in the graph of v_n can only occur at nodes.) As a result, we know that $v_n' = m$ on $[x_{i-1}, x_i]$ for some m. And so we want to show

$$\int_{x_{i-1}}^{x_i} (u - u_n)' \, m \, dx = \int_{x_{i-1}}^{x_i} (u_I - u_n)' \, m \, dx$$

We're done by the Fundamental Theorem of Caclulus:

$$\int_{x_{i-1}}^{x_i} (u - u_n)' \, m \, dx = m \left[u(x_i) - u_n(x_i) - u(x_{i-1}) + u_n(x_{i-1}) \right]$$
$$= m \left[u_I(x_i) - u_n(x_i) - u_I(x_{i-1}) + u_n(x_{i-1}) \right] = \int_{x_{i-1}}^{x_i} (u_I - u_n)' \, m \, dx$$

Above, I used that u and u_I agree at the nodes. This finishes the proof.

We now show that the approximation assumption (2) holds for this space. Because the left-hand side of (2) is an infimum taken over all piecewise linear functions v it suffices to show that the inequality holds for a particular piecewise linear functions. We do this by showing the inequality holds if we take the linear interpolant, w_I, as a sample piecewise linear function.

Theorem: *If $w'' \in L^2$ and $h = \max\{x_j - x_{j-1}\}$ then*

$$\|w - w_I\|_E \leq \frac{h}{\sqrt{2}} \|w''\|$$

Therefore the approximation assumption (2) holds with $\epsilon = h/\sqrt{2}$.

Proof: Since

$$\|w - w_I\|_E^2 = \int_0^1 (w'(x) - w_I'(x))^2 \, dx = \sum_{j=1}^n \int_{x_{j-1}}^{x_j} (w'(x) - w_I'(x))^2 \, dx,$$

it suffices to prove the desired bound on each subinterval. And so we will prove

$$\int_{x_{j-1}}^{x_j} (w'(x) - w_I'(x))^2 \, dx \leq \tfrac{1}{2} \, (x_j - x_{j-1})^2 \int_{x_{j-1}}^{x_j} w''(x)^2 \, dx \qquad \forall \; 1 \leq j \leq n$$

First, we change variables, introducing

$$\tilde{x} = \frac{x - x_{j-1}}{x_j - x_{j-1}} \quad \Longrightarrow \quad \frac{d}{dx} = \frac{1}{x_j - x_{j-1}} \frac{d}{d\tilde{x}} \quad \text{and} \quad (x_j - x_{j-1}) \, d\tilde{x} = dx$$

In the new variables, we see it suffices to prove

$$\int_0^1 (w'(x) - w_I'(x))^2 \, dx \leq \tfrac{1}{2} \int_0^1 w''(x)^2 \, dx.$$

In the new coordinates, $w - w_I$ vanishes at both $x = 0$ and $x = 1$. Because $\int w''^2 < \infty$, we know that w' is continuous on $[0, 1]$. We also know that w_I' is continuous on $[0, 1]$ and therefore $w' - w_I'$ is too. Therefore, by the mean value theorem, there is some point ξ at which $w'(\xi) - w_I'(\xi) = 0$. Hence

$$w'(y) - w_I'(y) = \int_\xi^y w''(x) - w_I''(x) \, dx = \int_\xi^y w''(x) \, dx.$$

Above, we used that since w_I is linear, $w_I'' = 0$. By Schwartz's inequality,

$$\begin{aligned} \left| w'(y) - w_I'(y) \right| &= \left| \int_\xi^y w''(x) \, dx \right| \leq \sqrt{\int_\xi^y 1 \, dx} \sqrt{\int_\xi^y w''(x)^2 \, dx} \\ &= \sqrt{|\xi - y|} \sqrt{\int_\xi^y w''(x)^2 \, dx} \leq \sqrt{|\xi - y|} \sqrt{\int_0^1 w''(x)^2 \, dx} \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \left| w'(y) - w_I'(y) \right|^2 \, dy &\leq \int_0^1 |\xi - y| \, \int_0^1 w''(x)^2 \, dx \, dy \\ &= \int_0^1 w''(x)^2 \, dx \, \int_0^1 |\xi - y| \, dy \leq \tfrac{1}{2} \int_0^1 w''(x)^2 \, dx \end{aligned}$$

In the last step, we used that

$$\sup_{0<\xi<1} \int_0^1 |\xi - y| \, dy = \tfrac{1}{2}.$$

This finishes the proof.

Again, in the above proof I was quick on a few details. I constructed $w' - w_I'$ from w'' using the fundamental theorem of calculus. This holds only if $w' - w_I'$ is absolutely continuous. However I can get the desired bound via density arguments and the Lebesgue Dominated Convergence theorem, in a similar manner as I did in the proof of the Lax-Wendroff theorem.

We therefore know that for the subspace of linear interpolants the inequality (3) becomes

$$\|u - u_n\| \leq \frac{1}{\sqrt{2}} \max\{x_j - x_{j-1}\} \, \|u - u_n\|_E \leq \tfrac{1}{2} \left(\max\{x_j - x_{j-1}\}\right)^2 \|f\|$$

This shows that if we choose the nodes so that the maximum distance between nodes goes to zero as $n \to \infty$ then $u_n \to u$ in both the L^2 norm and in the energy norm.

2 How to compute $\langle f, \phi_j \rangle$

To find u_n we need to compute

$$\langle f, \phi_j \rangle = \int_0^1 f(x)\, \phi_j(x)\, dx = \int_{x_{j-1}}^{x_{j+1}} f(x)\, \phi_j(x)\, dx \qquad \forall\, j.$$

Rather than using a quadrature method in which we divide the interval $[x_{j-1}, x_{j+1}]$ into m subintervals and approximate the integral on each subinterval, we use Gaussian Quadrature.

Gaussian Quadrature approximates an integral by sampling the integrand at certain very well-chosen nodes and then making a weighted sum out of those samples:

$$\int_{-1}^{1} g(x)\, dx \approx \sum_{i=1}^{N} w_i \, g(x_i)$$

For example, if we use 5 sample points then the weights and sample points are

x_i	w_i
0	$128/225$
$\pm\sqrt{5 - 2\sqrt{10/7}}$	$(322 + 13\sqrt{70})/900$
$\pm\sqrt{5 + 2\sqrt{10/7}}$	$(322 - 13\sqrt{70})/900$

A simple change of variables allows us to use Gaussian Quadrature on an interval:

$$\int_a^b g(x)\, dx \approx \frac{b-a}{2} \sum_{i=1}^{N} w_i \, g\left(\frac{b-a}{2}\, x_i + \frac{a+b}{2}\right) \tag{5}$$

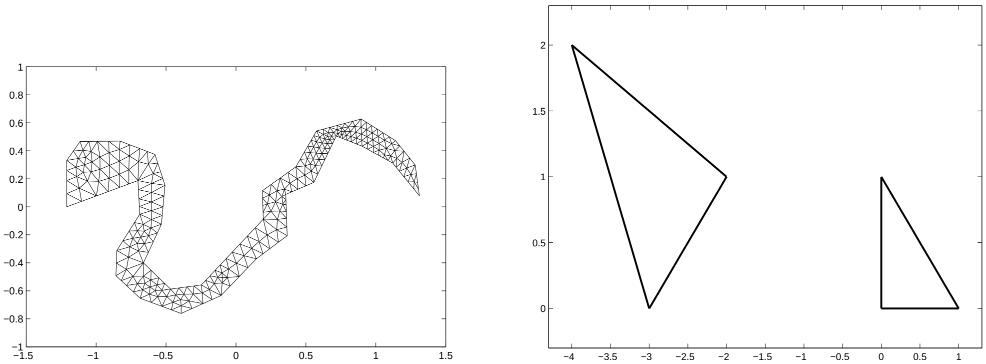

Figure 1: Left plot: a sample domain in the plane and triangularization. Right: the triangle on the left is an arbitrary triangle Δ. The triangle on the right is a reference triangle, Δ_0, for which good Gaussian Quadrature rules are known.

Given a domain in the plane, the first step is to construct a triangularization. See the left plot in Figure 1, for example. Similarly, given a domain in three-space, the first step is to decompose the domain into tetrahedrons. Basis functions are then defined and things proceed as in the one-dimensional example. To find

$$\int_\Delta f(x,y)\;\phi_j(x,y)\;dxdy$$

where Δ is an arbitrary triangle in the plane, we first change coordinates so that we need to compute

$$\int_{\Delta_0} \tilde{f}(x,y)\;\tilde{\phi}_j(x,y)\;dxdy$$

where Δ_0 is the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$. See Figure 1. To approximate the integral over Δ_0 we use a Gaussian Quadrature rule and sample at certain well-chosen points in Δ_0. This leads to a formula analogous to (5).