

Mat1062: Computational Methods for PDE

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1 Phase error

Another way to understand the error in the discrete computation is to consider solutions of the form $u(x, t) = U(t) \exp(i\xi x)$, for $\xi \in \mathbb{R}$. The true solution has $U(t) = \exp(-ia\xi t)$, so that $u(x, t) = \exp(i\xi(x - at))$. To relate these waves to the solutions of the discrete formula, we identify $x = jh$ and $t = nk$, and then see that in terms of the discrete dispersion relation $\eta(\omega)$

$$\omega = \exp(i\xi h), \quad \kappa = \frac{i}{k} \log \eta$$

so

$$\begin{aligned} \eta^n \omega^j &= \exp(-i\kappa nk) \exp(i\xi jh) = \exp(i(\xi x - \kappa t)) \\ &= \exp\left(i\xi \left(x - \frac{\operatorname{Re} \kappa}{\xi} t\right)\right) \exp(\operatorname{Im} \kappa \cdot t) \end{aligned}$$

The scheme would be perfect if $\kappa = a\xi$. Errors in the *imaginary* part of κ will correspond to anomalous growth or decay of the Fourier modes; for a stable scheme we will always have $|\eta| \leq 1$ so $\operatorname{Im} \kappa \leq 0$. Errors in the *real* part of κ correspond to incorrect propagation speed. Since the phase speed of the discrete mode is $c = \operatorname{Re} \kappa / \xi$, then $\operatorname{Re} \kappa > a\xi$ (for $\xi > 0$) means that the mode with wavenumber ξ is traveling too *fast*, whereas $\operatorname{Re} \kappa < a\xi$ means the mode is traveling too *slowly*.

As an example, for the upwind scheme we calculate

$$\begin{aligned} \kappa &= \frac{i}{k} \log\left(1 - \mu(1 - e^{-i\xi h})\right) \\ &\sim a\xi - i \frac{\mu(1 - \mu)}{2} \frac{h^2}{k} \xi^2 + \dots \end{aligned}$$

The first additional term is precisely the $-D\xi^2$ that we would expect for a diffusion, with $D = \frac{1}{2}\mu(1 - \mu)h^2/k$. The errors are *dissipative*.

And for the Lax-Wendroff scheme we have

$$\begin{aligned}\kappa &= \frac{i}{k} \log\left(1 - i\mu \sin \xi h - \mu^2(1 - \cos \xi h)\right) \\ &\sim a\xi \left(1 - \frac{1 - \mu^2}{6}(\xi h)^2 + \dots\right).\end{aligned}$$

Thus, modes with $\xi \neq 0$ travel slightly too *slowly*, by an amount that is second-order in h , but are not damped ($\text{Im } \kappa = 0$). The errors are *dispersive*. In practice, this can be even worse than dissipation since the maximum principle is not maintained. Note that the errors depend on the product ξh , the wavelength relative to the grid size, and is exactly zero if $a = 0$ or $\mu = 1$.