Mat1062: Computational Methods for PDE

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1 Ownership

These notes are built upon those of Rob Almgren who taught an analogous course in 2003. Whatever you learn of value from them is due to him. All mistakes and sources of confusion are to be blamed on me.

2 More on stability of discrete schemes

We will analyse the stability of a discrete scheme $P_{k,h}$ via Fourier methods. This approach was invented by the great John von Neumann.

2.1 Fourier Analysis

Recall that for a real- or complex-valued function $u(x)$ defined on $\mathbb{R}$ we can define the Fourier transform which is another function on $\mathbb{R}$

$$\widehat{u}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{-ix\xi} \, dx.$$

This is defined for a wide class of functions u, for simplicity's sake let's assume all our functions have finite L^1 norm: $\int |u(x)| \, dx < \infty$. The Fourier inversion formula states

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{u}(\xi) e^{i\xi x} \, d\xi.$$

Similarly, if one has a function $\{v_m\}$ defined on $\mathbb{Z}$ then its Fourier transform is

$$\widehat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} v_m e^{-im\xi}$$

which is defined for all ξ in $[-\pi, \pi]$ and $\widehat{v}(-\pi) = \widehat{v}(\pi)$. The Fourier inversion formula is

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \widehat{v}(\xi) e^{im\xi} \, d\xi$$

and Parseval's theorem is

$$\sum_{m=-\infty}^{\infty} |v_m|^2 = \int_{-\pi}^{\pi} |\widehat{v}(\xi)|^2 \, d\xi$$

If the function is defined on $h\mathbb{Z}$ then by a change of variables,

$$\widehat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} v_m \, e^{-ihm\xi} h,$$

for $\xi \in [-\pi/h, \pi/h]$ and the inversion formula is

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \widehat{v}(\xi) \, e^{ihm\xi} \, d\xi$$

and Parseval's theorem is

$$\|\widehat{v}\|_h^2 = \int_{-\pi/h}^{\pi/h} |\widehat{v}(\xi)|^2 \, d\xi = \sum_{m=-\infty}^{\infty} |v_m|^2 h = \|v\|_h^2$$

Note that if $\omega \in \mathbb{R}$ then $\omega = \xi + N2\pi/h$ for some $\xi \in [-\pi/h, \pi/h]$ and some $N \in \mathbb{Z}$. If we then sample the function $\exp(i\omega x)$ on $h\mathbb{Z}$ we find

$$e^{i\omega mh} = e^{i(\xi + N2\pi/h)\, mh} = e^{i\xi\, mh}.$$

In short, the grid $h\mathbb{Z}$ cannot "see" high frequencies ω such that $|\omega| > \pi/h$.

2.2 von Neumann Analysis

Let's return to the forward-time, forward-space scheme for $u_t + u_x = 0$:

$$\frac{u_m^{n+1} - u_m^n}{k} + \frac{u_{m+1}^n - u_m^n}{h} \quad \Longrightarrow \quad u_m^{n+1} = (1+\lambda) u_m^n - \lambda u_{m+1}^n$$

where $\lambda = k/h$. We know that this scheme is not convergent and so, by the Lax-Richtmyer theorem, it must be unstable. We write each of the terms in terms of its Fourier transform

$$u_m^{n+1} = \tfrac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \widehat{u}^{n+1}(\xi) \, e^{ihm\xi} \, d\xi$$
$$u_m^{n} = \tfrac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \widehat{u}^{n}(\xi) \, e^{ihm\xi} \, d\xi$$
$$u_{m+1}^{n} = \tfrac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} \widehat{u}^{n}(\xi) \, e^{ih(m+1)\xi} \, d\xi$$

hence

$$\int_{-\pi/h}^{\pi/h} \widehat{u}^{n+1}(\xi)\, e^{ihm\xi}\, d\xi = \int_{-\pi/h}^{\pi/h} \Big((1+\lambda) - \lambda e^{ih\xi}\Big)\, \widehat{u}^{n}(\xi)\, e^{ihm\xi}\, d\xi$$

from which we conclude

$$\widehat{u}^{n+1}(\xi) = \Big((1+\lambda) - \lambda e^{ih\xi}\Big)\, \widehat{u}^{n}(\xi) = \sigma_h(\xi)\widehat{u}^{n}(\xi), \qquad \text{for } -\pi \le h\xi \le \pi$$

and so we see that advancing the solution of the finite difference scheme by one step is the same as multiplying its Fourier transform by the factor $\sigma_h(\xi)$. Going all the way back to the initial data

$$\widehat{u}^{n}(\xi) = \sigma_h(\xi)^n\, \widehat{u_0}(\xi), \qquad \text{for } -\pi \le h\xi \le \pi.$$

We now analyse factor

$$\sigma_h(\xi) = 1 + \lambda - \lambda\cos(h\xi) - i\lambda\sin(h\xi),$$
$$\Longrightarrow \quad |\sigma_h(\xi)|^2 = 1 + 2\lambda(1+\lambda)(1-\cos(h\xi)) \ge 1$$

Indeed, this factor is greater than 1 for all $\xi \in [-\pi/h, \pi/h]$ except $\xi = 0$. You will show in a homework problem that for any finite-difference scheme if $|\sigma_h(\xi)| > 1$ then the scheme is not stable. Specifically, the forward-time, forward space scheme for $u_t + u_x = 0$ is not stable.

We now repeat this analysis for the three schemes for the diffusion equation $u_t = Du_{xx}$. First, we consider the explicit sheme

$$\frac{u_m^{n+1} - u_m^n}{k} = D\frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{k^2} \quad \Longrightarrow \quad \sigma_h(\xi) = 2\lambda\cos(h\xi) + 1 - 2\lambda$$

where $\lambda = kD/h^2$. Note that $\sigma_h(\xi)$ is an even function in ξ, it equals 1 at $\xi = 0$, and is decreasing in ξ on $[0, \pi/h]$. Its minimum value is at $h\xi = \pm\pi$

$$\max_{\xi\in[-\pi/h,\pi/h]} \sigma_h(\xi) = 1, \qquad \min_{\xi\in[-\pi/h,\pi/h]} \sigma_h(\xi) = 1 - 4\lambda$$

It follows immediately that if $\lambda > 1/2$ then the scheme is unstable. In Figure 1 you can see $\sigma_h(\xi)$ for a variety of values of λ.

We now consider the fully implicit scheme

$$\frac{u_m^{n+1} - u_m^n}{k} = D\frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{k^2} \quad \Longrightarrow \quad \sigma_h(\xi) = \frac{1}{1 + 2\lambda - 2\lambda\cos(h\xi)}$$

Again, $\sigma_h(\xi)$ is an even function in ξ, it equals 1 at $\xi = 0$, and is decreasing in ξ on $[0, \pi/h]$. Its minimum value is at $h\xi = \pm\pi$

$$\max_{\xi \in [-\pi/h, \pi/h]} \sigma_h(\xi) = 1, \qquad \min_{\xi \in [-\pi/h, \pi/h]} \sigma_h(\xi) = \frac{1}{1+4\lambda} > 0 > -1$$

In this way, we see that the scheme is stable for all λ and the factor $\sigma_h(\xi) > 0$ for all $\xi \in [-\pi/h, \pi/h]$.

Finally, we consider the Crank-Nicolson scheme

$$\frac{u_m^{n+1} - u_m^n}{k} = D \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{2k^2} + D \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2k^2}$$
$$\Longrightarrow \quad \sigma_h(\xi) = \frac{1 + \lambda\cos(h\xi) - \lambda}{1 + \lambda - \lambda\cos(h\xi)}$$

As above,

$$\max_{\xi \in [-\pi/h, \pi/h]} \sigma_h(\xi) = 1, \qquad \min_{\xi \in [-\pi/h, \pi/h]} \sigma_h(\xi) = \frac{1-2\lambda}{1+2\lambda} > -1$$

In this way, we see that the scheme is stable for all λ.

If you do the stability analysis of the θ-scheme

$$\frac{u_m^{n+1} - u_m^n}{k} = \theta D \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{k^2} + (1-\theta) D \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{k^2}$$

you will find that it's stable for all λ if $\theta \geq 1/2$. And that it's stable for $\lambda \leq \lambda_\theta$ otherwise.

3 Stability via ODE schemes

Here are notes that discuss stability by first analyzing the behaviour of certain one-step shemes for ODE. In the following, the "trapezoid" scheme is the Crank-Nicolson scheme applied to ODE.

The main idea is to look at how the solutions of the discrete scheme behave as $n \to \infty$ and $k \to 0$, with $nk = t$, *without reference to the true solution.* For a problem which is linear, or locally linear (as most are), this is the same as asking whether small changes in the initial data give rise to small changes in the solution.

Stability for discretizations of nonlinear ODEs is an extremely rich subject; there are several definitions of stability, with subtle differences among them. But since we are really interested in linear PDEs, let us make a

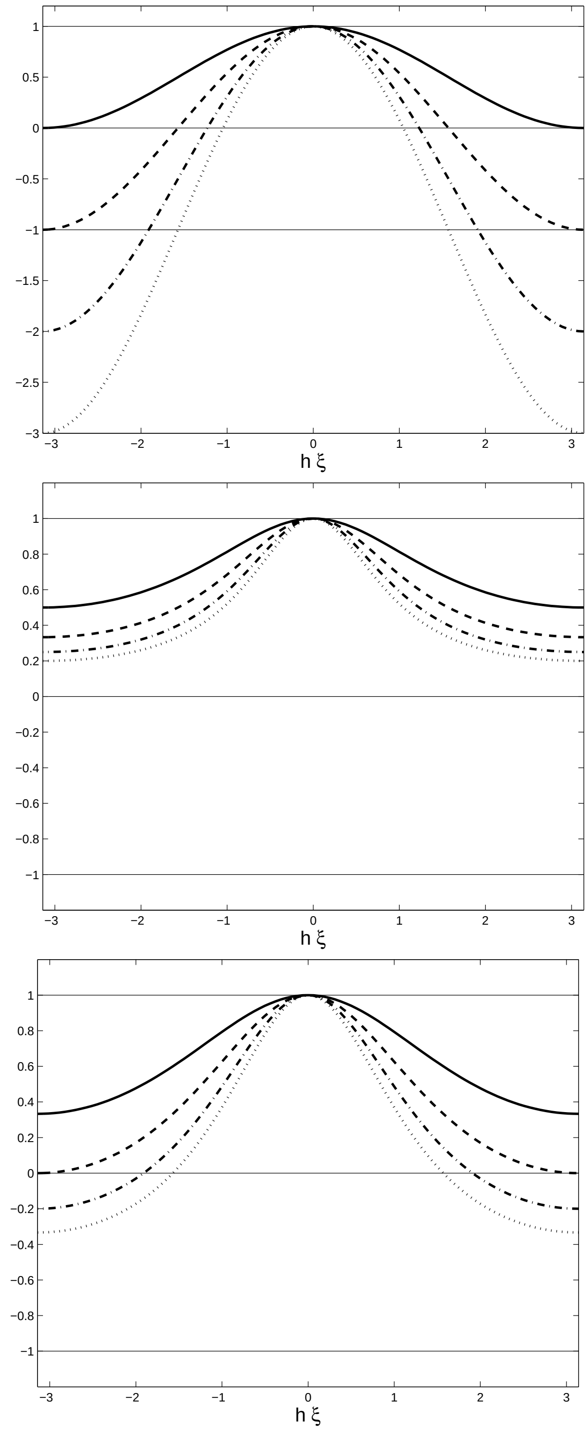

Figure 1: Top plot: forward Euler method, middle plot: backward Euler method, bottom plot: Crank-Nicolson method. All three plots show $\sigma_h(\xi)$ as a function of $h\xi$. Dotted line: $\lambda = 1$; dot-dash line: $\lambda = 3/4$, dashed line: $\lambda = 1/2$, solid line: $\lambda = 1/4$. In the plot for the forward Euler method, note that for $\lambda > 1/2$ there is always an interval of frequencies that will grow exponentially. In the plot for the backward Euler method, note that all frequencies dissipate monotonically in time ($\sigma_h(\xi) > 0$ for all ξ). In the plot for the Crank-Nicolson method, note that if $\lambda > 1/2$ the high frequencies do decrease in time but they oscillate as they do so. (There is "dispersive smoothing" if $\lambda > 1/2$.)

tremendous simplification. We shall consider only *linear* problems whose solutions *decay* in time. (Because are interested in the diffusion equation.) For our purposes now, let us say that a discrete method is *stable* if no solutions of the difference formula grow in time.

Suppose our system of ODE is

$$\frac{du}{dt} = Au + B,$$

where A and B are constant in time. Let us suppose that we have an eigenvalue decomposition $AX = X\Lambda$, where Λ is diagonal with entries $(\lambda_1, \ldots, \lambda_N)$, and we suppose that X is of full rank so that $A = X\Lambda X^{-1}$. Since we only are interested in problems whose solutions decay in time, we suppose that each $\lambda_j < 0$, and hence A is invertible.

Then we can define u in terms of a new vector y as $u = Xy - A^{-1}B$. Subsituting into the ODE, we find that y solves $dy/dt = \Lambda y$, which is a collection of independent scalar equations.

Thus let us consider only the simple linear scalar problem $f(u) = -\sigma u$, with $\sigma > 0$ (we take $\sigma = -\lambda$). The ODE is

$$u_t = -\sigma u,$$

whose exact solution is

$$u(t) = u_0 e^{-\sigma t}.$$

This is an exponential which decays on a time scale of $1/\sigma$.

Forward Euler

$$v^{n+1} = v^n - k\sigma v^n = (1 - k\sigma)\, v^n,$$

so the solution is

$$v^n = v^0 \eta^n, \qquad \text{with} \quad \eta = 1 - k\sigma.$$

The superscript on v^n is an index, while on η^n it is an exponent. By our definition, the scheme is stable if and only if $|\eta| \leq 1$, which requires

$$k \leq \frac{2}{\sigma}.$$

```
function ode( sigma, dt, tmax )
% ode( sigma, dt, tmax )  Plot discrete solution of ODE

nstep = floor(tmax/dt);
tmax = nstep*dt;        % make tmax be exact multiple of dt

%  Uncomment just one of these statements
eta = 1 - sigma*dt;                          % Forward Euler
%eta = 1 / ( 1 + sigma*dt );                  % Backward Euler
%eta = (1 - 0.5*sigma*dt)/(1 + 0.5*sigma*dt); % Trapezoid

t = linspace( 0, tmax, nstep+1 );
u = zeros(1,nstep+1);   % allocate storage for whole array
u(1) = 1;
for i=1:nstep; u(i+1) = eta*u(i); end

%  Exact exponential solution
t0 = linspace( 0, tmax, 101 );  u0 = exp(-sigma*t0);

plot( t, u, '-ok', t0, u0, 'k' ); xlabel('t'); ylabel('u(t)')
```

When σ is large, we need a small time step for stability. This is true for *all* explicit methods. The reason for the instability is clear (Figure 2, p. 9): When σ is large, the solution heads toward zero, but overshoots.

Setting $n = t/k$, we may also write

$$v^n = v^0 e^{-\sigma_k t},$$

where the decay rate for a solution computed with time step k is

$$\sigma_k = -\frac{1}{k}\log\eta = -\frac{1}{k}\log(1 - k\sigma) \;\sim\; \sigma + \mathcal{O}(\sigma^2 k), \quad \sigma k \to 0,$$

Solutions of the discrete problem approximate solutions of the continuous problem, as long as σk is small; that is, the the time step must be small relative to the intrinsic time scale of the solution itself. This is always true for explicit methods.

Backward Euler

$$v^{n+1} = v^n - k\sigma v^{n+1}, \qquad \text{so} \qquad v^{n+1} = \frac{1}{1 + k\sigma} v^n$$

Again $v^n = v^0\eta^n$, but now

$$\eta = \frac{1}{1+k\sigma}$$

(see Figure 5). Since $\eta < 1$ for *all* $k > 0$ (for $\sigma > 0$), the scheme is *stable for all timesteps* k. This is a characteristic of well-constructed implicit methods, and the reason for their use. Now the discrete growth rate is

$$\sigma_k = -\frac{1}{k}\log\frac{1}{1+k\sigma} \;=\; \frac{1}{k}\log(1+k\sigma) \;\sim\; \sigma + \mathcal{O}(\sigma^2 k), \quad \sigma k \to 0.$$

Since $\sigma > \sigma_k > 0$, solutions to the backward Euler formula always decay in time, though not as fast as the continuous solution. In order for the discrete solution to approximate the continuous solution, we still need $k < 1/\sigma$.

When k is large for a given σ, η is very near zero, so the discrete solution decays very rapidly. The discrete solution "fails gracefully."

Trapezoid

$$v^{n+1} = v^n - \tfrac{1}{2}k\sigma(v^{n+1} + v^n),$$

so $v^n = v^0\eta^n$ with

$$\eta = \frac{1-\frac{1}{2}k\sigma}{1+\frac{1}{2}k\sigma}$$

and now

$$\sigma_k = \sigma + \mathcal{O}(k^2), \quad k \to 0.$$

The method is again stable for *all* $k > 0$ (Figure 5). For large σk, η is near -1; the solutions oscillate in time rather than decaying (Figure 4, p. 11). Nonetheless, they do not grow. The approximation is one higher order in k near $k = 0$ than with Euler, since this is a second-order method.

Leapfrog

$$v^{n+1} = v^{n-1} - 2k\sigma v^n$$

Because this is a two-level formula, the discrete system has *two* exponential solutions. Looking for solutions in the form $v^n = \eta^n$, we get the quadratic

$$\eta^2 + 2k\sigma\eta - 1 = 0,$$

whose solutions are

$$\eta = -k\sigma \pm \sqrt{(k\sigma)^2 + 1}.$$

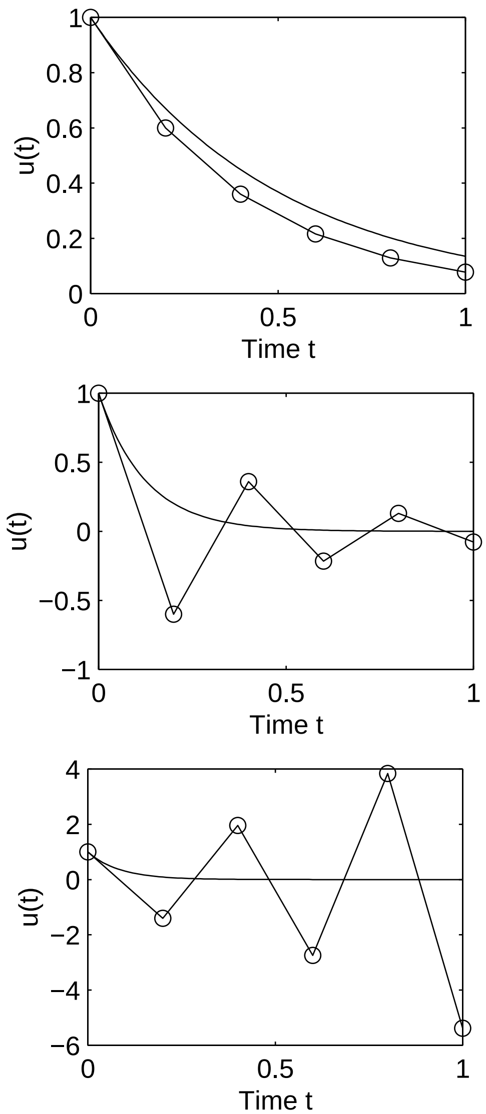

Figure 2: Forward Euler method with $\sigma = 2, 8, 12$, $k = 0.2$. For $\sigma k < 1$, the discrete solution is well-behaved. For $1 < \sigma k < 2$, the discrete solution has $-1 < \eta < 0$ (see Figure 5, so it decays with oscillations. For $\sigma k > 2$, $\eta < -1$, so the discrete solution oscillates and grows.

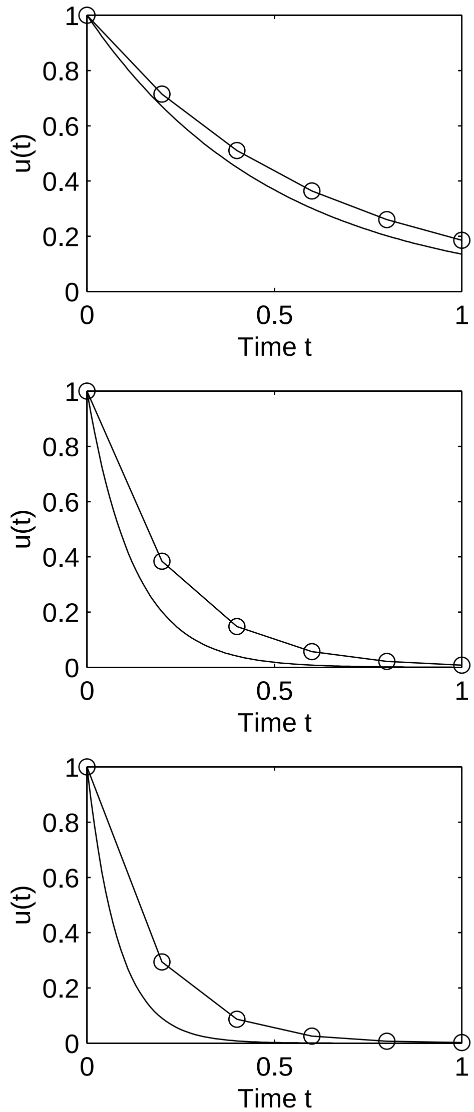

Figure 3: Backward Euler method with $\sigma = 2, 8, 12$, $k = 0.2$. For all values of σk, we have $0 < \eta < 1$, so the solution always decays without oscillating. Since $\eta > \eta_{\text{true}}$, the discrete solution decays more slowly than the true solution.

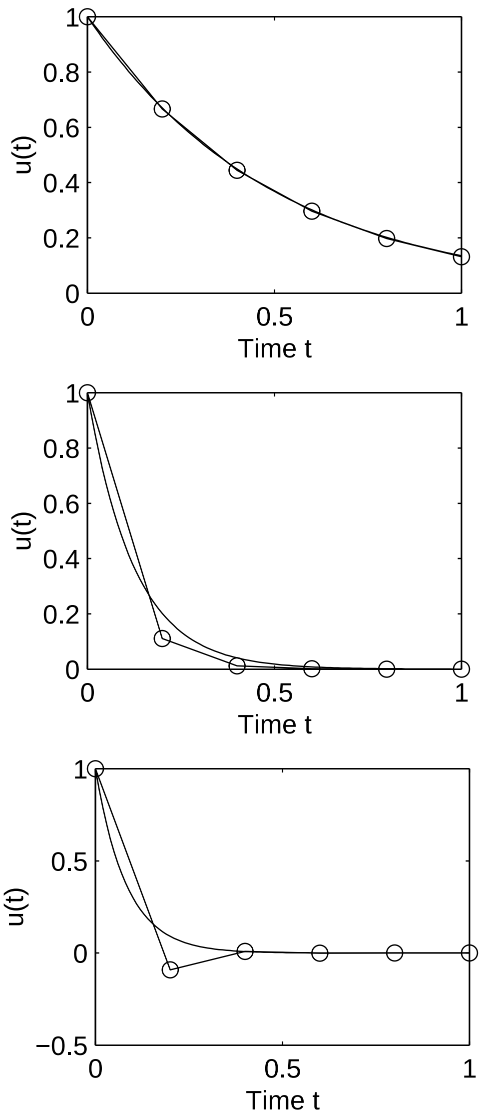

Figure 4: Trapezoid method with $\sigma = 2, 8, 12$ and $k = 0.2$. The discrete solution is well-behaved for all values of σk since $|\eta| < 1$, though it exhibits mild oscillations for $\sigma k > 2$, when $\eta < 0$. For small σk it gives an extremely good approximation (second-order accurate).

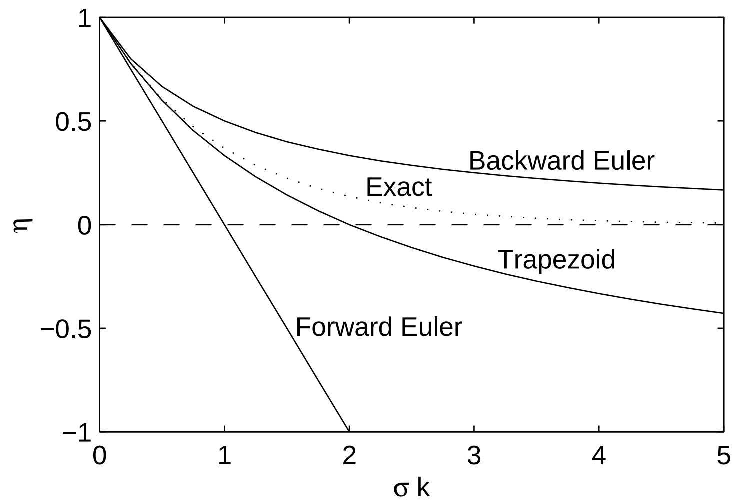

Figure 5: Amplification factors η as functions of σk, and $\eta_{true} = e^{-\sigma k}$. We must have $|\eta| \leq 1$ for stability. The forward Euler method loses stability when $\sigma k > 2$. The trapezoid (Crank-Nicolson) method has a negative amplification factor for $\sigma k > 2$, but it is stable for all $\sigma k > 0$. It has a second-order accurate match to the true solution near $\sigma k = 0$.

The general solution to the difference formula is

$$v^n = C_+ \eta_+^n + C_- \eta_-^n$$

where $C_\pm$ are determined by the necessary two levels of initial data, and

$$\eta_+ = -k\sigma + \sqrt{(k\sigma)^2 + 1} \sim 1 - k\sigma + \ldots, \quad k \to 0$$
$$\eta_- = -k\sigma - \sqrt{(k\sigma)^2 + 1} \sim -1 - k\sigma + \ldots, \quad k \to 0.$$

For all $k > 0$ (and $\sigma > 0$), $|\eta_+| < 1$ and $|\eta_-| > 1$ (you can see this by considering a right triangle whose sides are 1, $k\sigma$, and $\sqrt{(k\sigma)^2 + 1}$). The solution η_+^n is a good approximation to the solution of the ODE. The solution η_-^n is a artificial, unstable, solution of the difference formula. Since the general solution includes both modes, that is, small perturbations will make both C_+ and C_- nonzero, the scheme is *unstable for all* k.

All two-level schemes have two discrete solutions, one physical and one unphysical. But for a well-constructed method, the unphysical solution *decays* in time and does not affect the accuracy of the solution.

4 Stability for PDEs

Because our PDEs are linear, their general solutions are combinations of simple elementary solutions. For stability analysis of difference schemes, the relevant solutions are *eigenvectors:* solutions which have simple exponential behavior in time. For a constant-coefficient linear PDE, the eigenvectors are the Fourier modes $e^{i\xi x}$ where ξ is real. (If you don't like complex numbers, you can just think of $\sin \xi x$ and $\cos \xi x$ instead.)

The Fourier modes have the form

$$u(x,t) = A(\xi,t)e^{i\xi x},$$

where, on an unbounded domain, ξ may be any real number. Substituting into the PDE $u_t = Du_{xx}$, we see that we need

$$A_t = -D\xi^2 A$$

so

$$A(t) = A(\xi,0)\, e^{-\sigma t},$$

where the *continuous dispersion relation* is

$$\sigma(\xi) \;=\; D\xi^2. \tag{1}$$

In general, for a linear equation, the term "dispersion relation" refers to the relation giving the time behavior σ in terms of the spatial wave number ξ. In this case it tells us that short-wavelength modes (large ξ) decay rapidly (large positive σ).

Boundary conditions restrict the set of allowable ξ. For example, if the domain has length L with Dirichlet boundary conditions, then the solution must contain only modes of the form $\sin(n\pi x/L)$, corresponding to $\xi = n\pi/L$ for $n = 0, 1, \ldots$.

4.1 Space discretization

On a spatially discrete grid with step h, our *semi-discrete* approximation is the collection of ODEs

$$\frac{dU_j}{dt} \;=\; \frac{D}{h^2}\Big(U_{j+1} - 2U_j + U_{j-1}\Big),$$

with appropriate modifications at the endpoints to include the boundary conditions.

We may again analyze this system using Fourier analysis. Consider a single Fourier mode with wave number ξ

$$U_j(t) \;=\; A(t)\, e^{i\xi x_j} \;=\; A(t)\, e^{i\xi j h} \;=\; A(t)\,\omega^j, \qquad \text{with } \omega = e^{i\xi h}.$$

Since the discrete solution should be bounded in space, we need ξ real and $|\omega| = 1$ (an imaginary component of ξ would correspond to $|\omega| < 1$ or $|\omega| > 1$, and exponential growth as x and j either increase or decrease). Again, boundary conditions restrict the set of allowable ξ, but do not change the overall picture.

Then we readily compute

$$\frac{D}{h^2}\Big(U_{j+1} - 2U_j + U_{j-1}\Big) \;=\; \frac{D}{h^2}\, A(t)\,\omega^j \left(\omega - 2 + \frac{1}{\omega}\right) \;=\; -D\, A(t)\,\omega^j\, \frac{2(1-\cos\xi h)}{h^2},$$

since $|\omega| = 1$, $\omega^{-1} = \overline{\omega}$ and $\omega + \omega^{-1} = 2\,\mathrm{Re}\,\omega = 2\cos\xi h$. Also,

$$\frac{dU_j}{dt} \;=\; A'(t)\,\omega^j.$$

Equating these two expressions, our system is equivalent to the ODE

$$A'(t) \;=\; -2D\, \frac{1-\cos\xi h}{h^2}\, A(t), \qquad \text{so} \qquad A(t) \;=\; A(0)\, e^{-\sigma_h t}.$$

The *discrete dispersion relation* is

$$\sigma_h(\xi) \;=\; 2D\, \frac{1-\cos\xi h}{h^2}. \tag{2}$$

This should be compared with the continuous version (1); see Figure 6.

For small ξh,

$$\cos\xi h \;\sim\; 1 \;-\; \tfrac{1}{2}\xi^2 h^2 \;+\; \mathcal{O}\Big(\xi^4 h^4\Big),$$

so

$$\sigma_h(\xi) \;\sim\; \sigma(\xi) \;+\; \mathcal{O}\Big(\xi^4 h^2\Big), \qquad \xi h \to 0.$$

Now, the wavelength of our Fourier mode is $\ell = 2\pi/\xi$. So $\xi h = 2\pi h/\ell$, and the discrete model is a good approximation to the continuous problem when the grid spacing h is small compared to the wave length ($\xi h \ll 1$).

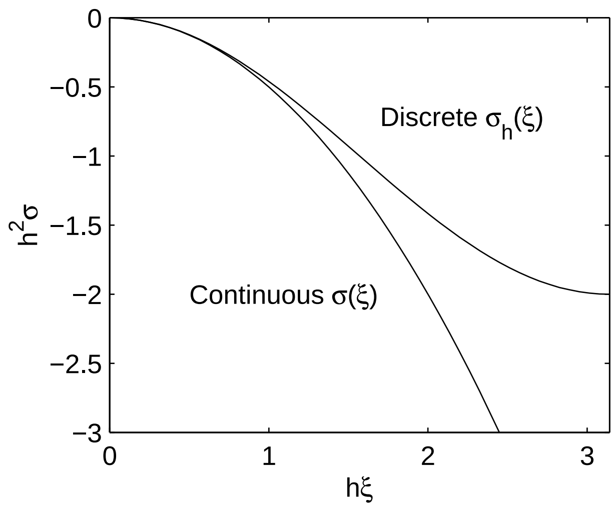

Figure 6: Continuous and discrete amplification factors $\sigma(\xi)$ and $\sigma_h(\xi)$.

How large can ξh be? The highest wavenumber that can be represented on a discrete grid has wavelength equal to twice the grid spacing, so that alternate grid points have values ± 1. (Higher wavenumbers are mapped back to lower ones by *aliasing.*) Thus the maximum value is $\xi h = \pi$. For $0 < \xi \le \pi/h$, the discrete decay rate $\sigma_h(\xi)$ satisfies $\sigma(\xi) < \sigma_h(\xi) < 0$ (see picture). So in the semi-discrete model, all modes decay though not as rapidly as for the continous PDE. Since $\sigma_h(\xi)$ is a decreasing function, it takes its most negative value at $\xi = \pi/h$, and this extreme value is

$$\sigma_{\max}(h) \;=\; \frac{4D}{h^2}. \tag{3}$$

We can now use our understanding of the ODE system to predict the stability of the discrete methods for the PDE $u_t = Du_{xx}$; we simply use $\sigma_{\max}$ for σ. The parameter that controls stability is clearly

$$\lambda \;=\; \tfrac{1}{4}\sigma_{\max}k \;=\; \frac{Dk}{h^2}.$$

- The **forward Euler** method is stable for $\sigma_{\max}k \le 2$; that is, for $\lambda \le \frac{1}{2}$. If $\lambda > \frac{1}{2}$, then the instability will be a "checkerboard" of alternate-grid-point oscillations in both space and time, growing exponentially in time. For $1 < \sigma_{max} k \leq 2$, or $\frac{1}{4} < \lambda \leq \frac{1}{2}$, it will display oscillations on successive time steps, but these oscillations will decay in time.

Recall that for the point mass solution, we had identified $\lambda = \frac{1}{2}$ as the threshhold value at which the solution ceased to be positive. Now we see that in fact the whole scheme is completely *unstable* for $\lambda > \frac{1}{2}$. Thus the point mass analysis was rather misleading.

- The **backwards Euler** method is stable for all ratios of k and h^2. There will never be any oscillation in time.

- The **Crank-Nicolson** (trapezoid) method is stable for all ratios of k and h^2. For $\sigma_{max} k > 2$, or $\lambda > \frac{1}{2}$, it will display oscillations on alternate time steps, but these oscillations will decay in time. (They are especially visible near a discontinuity in the initial data.)

For the forward Euler method for the ODE, we could always make the scheme stable by decreasing k until $\sigma k < 2$. For a PDE, σ depends on h, so as we decrease both h and k we may never be stable. If we choose $k = \lambda h^2/D$ with λ constant as $h, k \to 0$, then we will always be stable or always be unstable depending on whether $\lambda < \frac{1}{2}$ or $\lambda > \frac{1}{2}$. If we choose $k = \mu h$ for *any* fixed μ, then the scheme will always be unstable when k and h are small, since $Dk/h^2 = D\mu/h \to \infty$ as $h \to 0$.

For $\theta \geq \frac{1}{2}$, the scheme will be stable for *any* chosen relationship between k and h; the linear scaling $k = \mu h$ is often a convenient choice.