Mat1062: Computational Methods for PDE

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January 17, 2008

1 Ownership

These notes are built upon those of Rob Almgren who taught an analogous course in 2003. Whatever you learn of value from them is due to him. All mistakes and sources of confusion are to be blamed on me.

2 Convergence, Consistency, and Stability

Ordinary differential equations (ODEs) are fundamental models for time evolution problems. An ODE has the form

$$\frac{dU}{dt} = f\big(U(t), t\big) \quad \text{for } t \in [0, T], \qquad \text{with} \quad U(0) = U_0.$$

Here $U(t)$ is an element of $\mathbb{R}^N$, and f is a function from $\mathbb{R}^N \times [0, t]$ into $\mathbb{R}^N$. This formulation[1] generalizes naturally to partial differential equations, if we consider U to be an element in some function space, and f to be a linear or nonlinear operator acting on that space. We name the type of the ODE depending on properties of the function f.

The equation is *linear* if $f(U, t) = A(t)\, U + B(t)$, that is, if f is "affine linear" in U. In general, $A(t)$ is a matrix, and $B(t)$ is a vector. It is *linear and homogeneous* if $f(U, t) = A(t)\, U$, that is, $B = 0$ so f is linear in U.

If $f(U, t) = f(U)$, independent of t, then the equation is *autonomous*. If $f(U, t) = f(t)$, independent of U, then the ODE is an indefinite integral: $U(t) = U_0 + \int_0^t f(s)\, ds$.

[1] Note that everything could be done just as well for complex-valued ODE for which all the vectors are in $\mathbb{C}^N$.

Choose an integer M and a final time T, thus determining the time step $k = T/M > 0$. Define time "levels" $t_n = nk$, for $n = 0, 1, \ldots, M$. We want to construct a sequence of points in $\mathbb{R}^N$: $u^0, u^1, \ldots, u^M$, so that $u^n \approx U(nk)$ when k is small. Here u^n is the solution of the fully discrete model, and $U(nk)$ is the exact solution of the ODE, evaluated at the discrete time points.

We need a scheme for constructing the discrete approximation; such a scheme must consist of two things. First, we need a rule for picking the initial value u^0. Since $U(t)$ and u^n are members of the same space, we can usually just take $u^0 = U(0)$. Second, we need a rule of the form $u^{n+1} = F(u^n, \ldots, u^0; k)$ for computing each step of the approximation in terms of earlier steps. In fact, F could well be different for each step n.

A rich class of schemes are the "linear multistep formulas," for which u^{n+1} is a linear function of $u^n, \ldots, u^0$, $f^n, \ldots, f^0$, and possibly f^{n+1}, where $f^n = f(u^n, t_n)$:

$$u^{n+1} = \sum_{j=0}^{p} a_j u^{n-j} + k \sum_{j=-1}^{p} b_j f^{n-j}$$

where $p > 0$ and $a_0, \ldots a_p$ and $b_{-1}, \ldots b_p$ are constants. If either a_p or b_p is nonzero this is called a $p+1$-multistep method. If $b_{-1} = 0$ the scheme is *explicit*. Otherwise, it is an *implicit* scheme.

Note that a 1-step scheme needs only one initial value: given u^0 one can define u^1 which one can then use to define u^2 which one can then use to define u^3 and so on. For a p-step scheme with $p > 1$ one needs p values (u^0, $\ldots u^{p-1}$) to get the process started. One usually does this with a sequence of q-step regimes where q increases from 1 to $p-1$. For example, if one had a 3-step scheme, say, what one usually does is: given u^0 one uses a 1-step scheme to define u^1. One then uses a 2-step scheme to define u^2 from u^0 and u^1 and one can then use the 3-step scheme from then on.

Given a time-stepping scheme for an ODE there are three immediate questions: is the scheme consistent? is it convergent? is it stable? For multistep schemes for nonlinear ODEs, these questions are rather tricky. They are much simpler for 1-step methods applied to linear PDEs, and so we now restrict our attention to this case. Much of the following comes from sections 1.4 and 1.5 of Strikwerda's "Finite Difference Schemes and Partial Differential Equations" which is on reserve in the library.

2.1 Convergence

Consider a linear partial differential equation of the form

$$P(\partial_t, \partial_x)u(x,t) = f(x,t) \tag{1}$$

which is of first order in the derivative with respect to t. Examples would include:

$$u_t - bu_{xx} + au_x = 0$$
$$u_t - cu_{txx} + bu_{xxxx} = 0$$
$$u_t + cu_{tx} + au_x = 0$$

Consider a fixed one-step finite difference scheme for the equation

$$P_{k,h}u = f. \tag{2}$$

Given initial data $\{u_m^0\}$ where $m \in \mathbb{Z}$ we denote the solution of the scheme (2) by u_m^n where m ranges over $\mathbb{Z}$ and n ranges over $\mathbb{N}$.

Let $u(x,t)$ be the solution of the PDE (1) with initial data $u_0(x)$. Given a spatial grid spacing h, generate approximate initial data $\{u_m^0\}$ in such a way that as mh converges to x the approximate initial data u_m^0 converges to $u_0(x)$. For each fixed h and k, let u_m^n be the resulting solution of the one-step finite difference scheme (2). The scheme is convergent if u_m^n converges to $u(x,t)$ as (mh, nk) converges to (x,t) as h and k converge to 0.

Usually, the approximate initial data is taken to be $u_0(mh)$ — just sample the function at the grid points. In which case the first convergence statement is immediate if u_0 is continuous. Note that this "definition" of convergence is somewhat squishy because I haven't rigorously specified what I mean by a function on a grid (u_m^n) converging to a function defined on a semi-infinite strip ($u(x,t)$).

2.2 Consistency

In practice, proving that a scheme is convergent is not easy. However, if your scheme is "consistent" and "stable" then it will be convergent. And checking consistency and stability isn't hard. We start by defining what "consistency" means.

Given a partial differential equation, $Pu = f$, *and a finite difference scheme,* $P_{k,h}u = f$, *we say the finite difference scheme is* *consistent* *with the partial differential equation if for any smooth* $\phi(x,t)$

$$P\phi - P_{k,h}\phi \to 0, \qquad \text{as } k, h \to 0,$$

the convergence being pointwise convergence at each grid point.

Consider the one-way wave equation $u_t + u_x = 0$ for which

$$P = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$$

and the difference operator $P_{k,h}$ given by the forward-time forward-space sheme

$$P_{k,h}\phi = \frac{\phi_m^{n+1} - \phi_m^n}{k} + \frac{\phi_{m+1}^n - \phi_m^n}{h}$$

Here, ϕ is a smooth function and $\phi_m^n = \phi(mh, nk)$. We approximate all the terms in $P_{k,h}\phi$ with Taylor series in x and t about the point $(x_m, t_n) = (mh, nk)$.

$$\begin{aligned} \phi_m^{n+1} &= \phi_m^n + k\phi_t + \tfrac{1}{2}k^2\phi_{tt} + O(k^3) \\ \phi_{m+1}^n &= \phi_m^n + h\phi_x + \tfrac{1}{2}h^2\phi_{xx} + O(h^3) \end{aligned}$$

Plugging this in,

$$P_{k,h}\phi = \phi_t + a\phi_x + \tfrac{1}{2}k^2\phi_{tt} + \tfrac{1}{2}h\phi_{xx} + O(k^2) + O(h^2)$$

where the derivatives are all evaluated at (x_m, t_n). Thus

$$P\phi - P_{k,h}\phi = -\tfrac{1}{2}k^2\phi_{tt} - \tfrac{1}{2}h\phi_{xx} + O(k^2) + O(h^2) \quad \to \quad 0$$

as $(h,k) \to (0,0)$. Therefore this scheme is consistent.

Above, you have to think things through a little carefully. Naively, if $(x_m, t_n) = (mh, nk)$ where m and n are fixed then $h, k \to 0$ would force $x_m, t_n \to 0$. This isn't what's intended above. What you should understand is: Let $\phi(x,t)$ be a smooth function on $\mathbb{R} \times [0, \infty)$. Pick (x_0, t_0) in that region. Without loss of generality, assume $x_0 > 0$. For each n and m in $\mathbb{N}$, take $h_m := x_0/m$ and $k_n := t_0/n$. By construction, $(x_0, t_0) = (mh_m, nt_n)$ and

$$\begin{aligned} P\phi(x_0,t_0) - P_{k,h}\phi(x_0,t_0) = -\tfrac{1}{2}k^2\phi_{tt}(x_0,t_0) - \tfrac{1}{2}h\phi_{xx}(x_0,t_0) \\ +O(1/n^2) + O(1/m^2) \quad \to \quad 0 \end{aligned}$$

as $m, n \to \infty$. Note that m and n have to go to infinity simultaneously. Also note that the definition of consistency had nothing to do with ϕ being a solution of the PDE. It had to do with how the approximation of the operator acts on smooth functions.

Consistency implies that the solution of the partial differential equation, if it is smooth, is an approximate solution of the finite difference scheme. Similarly, convergence means that a solution of the finite difference scheme approximates a solution of the partial differential equation. It is natural to ask whether consistency is sufficient for a scheme to be convergent. Consistency is certainly necessary for convergence, but as the following example shows,a scheme may be consistent but not convergent.

Take the above forward-time forward-space scheme for the PDE $u_t + u_x = 0$. You can rewrite it as

$$u_m^{n+1} = u_m^n - \frac{k}{h}\left(u_{m+1}^n - u_m^n\right) = (1+\lambda)u_m^n - \lambda u_{m+1}^n$$

where $\lambda = k/h$. Now, take initial data

$$u_0(x) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{otherwise} \end{cases}$$

The solution is $u(x,t) = u_0(x-t)$ and so for any $t > 0$ we have $u(x,t) = 1$ if $x < t$. And so, if $t > 0$ there are positive x at which $u(x,t) > 0$. For the difference scheme, take initial data

$$u_m^0 = \begin{cases} 1 & \text{if } m < 0 \\ 0 & \text{otherwise} \end{cases}$$

Because $u_m^{n+1} = (1+\lambda)u_m^n - \lambda u_{m+1}^n$ it is clear that if $m > 0$ then $u_m^0 = 0$ and $u_m^n = 0$ for all $n > 0$. This is because u_m^{n+1} is determined by data below and to the right.

Clearly, if $x_m > 0$ it is impossible for u_m^n to converge to $u(x_m, t_n)$ if as $h, k \to 0$ and the scheme is not convergent.

2.3 Stability

We now define what stability means for a *homogeneous* finite difference scheme

A finite difference scheme $P_{k,h}u = 0$ for a first-order equation is stable if there is an integer J and positive numbers h_0 and k_0 such that for any positive time T there is a constant C_T such that

$$h \sum_{m=-\infty}^{\infty} |u_m^n|^2 \leq \widetilde{C}_T \sum_{j=0}^{J} h \sum_{m=-\infty}^{\infty} |u_m^j|^2$$

for $0 \leq nk \leq T$, $0 < h \leq h_0$, and $0 < k \leq k_0$.

Note that if we introduce the discrete analogue of the $L^2(\mathbb{R})$ norm

$$||w||_h = \left(h \sum_{m=-\infty}^{\infty} |w_m|^2 \right)^{1/2}$$

then the stability constraint can be written as

$$||u^n||_h^2 \leq \widetilde{C}_T \sum_{j=0}^{J} ||u^j||_h^2$$

and hence

$$||u^n||_h \leq C_T \sum_{j=0}^{J} ||u^j||_h \tag{3}$$

for some C_T.

We will usually use Fourier methods to check the stability of a scheme rather than trying to show directly that inequalty (3) holds. But here is a simple example where we can show it directly. Consider the scheme

$$u_m^{n+1} = \alpha u_m^n + \beta u_{m+1}^n$$

We show that if $|\alpha| + |\beta| \leq 1$ then this scheme is stable.

$$
\begin{aligned}
h \sum_{m=-\infty}^{\infty} |u_m^{n+1}|^2 &= h \sum_{m=-\infty}^{\infty} |\alpha u_m^n + \beta u_{m+1}^n|^2 \\
&= h \sum_{m=-\infty}^{\infty} |\alpha|^2 |u_m^n|^2 + 2|\alpha||\beta||u_m^n||u_{m+1}^n| + |\beta|^2 |u_{m+1}^n|^2 \\
&\leq h \sum_{m=-\infty}^{\infty} |\alpha|^2 |u_m^n|^2 + |\alpha||\beta|(|u_m^n| + |u_{m+1}^n|)^2 + |\beta|^2 |u_{m+1}^n|^2 \\
&= h \sum_{m=-\infty}^{\infty} (|\alpha| + |\beta|)^2 |u_m^n|^2 \leq h \sum_{m=-\infty}^{\infty} |u_m^n|^2
\end{aligned}
$$

One can repeat this argument all the way down to 0, resulting in

$$
h \sum_{m=-\infty}^{\infty} |u_m^{n+1}|^2 \leq h \sum_{m=-\infty}^{\infty} |u_m^0|^2
$$

and so $C_T = 1$ and $J = 0$ works for the stability inequality (3).

The stability inequality (3) states that the L^2 norm of the discrete solution at level n is bounded by some constant times the sum of the L^2 norms of the discrete solution at the first $J + 1$ levels. For one-step schemes, we will see that one can take $J = 0$ — all that matters is the L^2 norm of the discrete initial data.

From your PDE course, you may recall that there are various definitions of what it means for an initial value problem to be well-posed. If one considers linear, homogenous problems then L^2 well-posedness would be defined as:

The initial value problem for the first-order partial differential equation $Pu = 0$ *is well-posed if for any time* $T > 0$ *there is a constant* C_T *such that any solution* $u(x, t)$ *satisfies*

$$
\sqrt{\int_{-\infty}^{\infty} |u(x,t)|^2 \, dx} \leq C_T \sqrt{\int_{-\infty}^{\infty} |u(x,0)|^2 \, dx} \tag{4}
$$

for $0 \leq t \leq T$.

Indeed, if you can show that the inequality (4) holds then uniqueness of solutions and continuous dependence of solutions on initial data follows immediately.

It turns out that if the finite difference scheme is related to a PDE which is well-posed then consistency, convergence, and stability are all closely related:

Lax-Richtmyer Equivalence Theorem *A consistent one-step scheme for a well-posed initial value problem for a partial differential equation (1) is convergent if and only if it is stable.*