Homework problems: September 25, 2007

1. Let $u(x, y, t)$ be a solution of the heat equation $u_t = \alpha^2(u_{xx} + u_{yy})$. Let U be the function defined by

$$U(X(x, y), Y(x, y), t) = u(x, y, t)$$

   (a) If $X(x, y) = x + 3$ and $Y(x, y) = y - 1$, what PDE does $U(X, Y, t)$ satisfy? Give a physical explanation for what you found.

   (b) If $X(x, y) = -x$ and $Y(x, y) = y$, what PDE does $U(X, Y, t)$ satisfy? Give a physical explanation for what you found. If the initial data is even in x about $x = 0$ what can you say about the solution $u(x, y, t)$? If the initial data is odd in x about $x = 3$ what can you say about the solution $u(x, y, t)$? What if the initial data is even in y about $y = 0$?

   (c) If

$$X(x, y) = \frac{\sqrt{3}}{2}x - \frac{1}{2}y, \quad \text{and} \quad Y(x, y) = \frac{1}{2}x + \frac{\sqrt{3}}{2}x,$$

   what PDE does $U(X, Y, t)$ satisfy? Give a physical explanation for what you found. If the initial data is rotationally symmetric about $(x, y) = (0, 0)$ then what can you say about the solution $u(x, y, t)$?

   (d) If

$$X(x, y) = 2x, \quad Y(x, y) = 2y, \text{and} \quad T(t) = 4t$$

   what PDE does $U(X, Y, T)$ satisfy? Give a physical explanation for what you found.

2. In solving the heat equation via separation of variables, Farlow is a bit shallow at the the following point. He's found that

$$\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = k.$$

This implies that $T(t)$ and $X(x)$ satisfy the ODEs

$$T'(t) = k\alpha^2 T(t), \qquad X''(x) = kX(x).$$

If the constant k is a negative number then one can represent k as $-\lambda^2$ and the ODEs become

$$T'(t) = -\lambda^2\alpha^2 T(t), \qquad X''(x) = -\lambda^2 X(x),$$

resulting in the solution $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$ for constants A and B and a $T(t)$ that decays to zero as $t \to \infty$.

If the constant k is zero then the ODEs become

$$T'(t) = 0, \qquad X''(x) = 0,$$

resulting in the solution $X(x) = Ax + B$ for constants A and B and a $T(t)$ that is constant.

If the constant k is a positive number then one can represent k as λ^2 and the ODEs become

$$T'(t) = \lambda^2\alpha^2 T(t), \qquad X''(x) = \lambda^2 X(x),$$

resulting in the solution $X(x) = A\cosh(\lambda x) + B\sinh(\lambda x)$ for constants A and B and a $T(t)$ that goes to infinity as $t \to \infty$.

The recommended text, DuChateau & Zachmann, has a good discussion of these Sturm-Liouville problems.

The point of the following problem is for you to make sure that Farlow didn't miss anything in the cases of "clamped" boundary conditions, no-flux boundary conditions, or the boundary conditions that correspond to the rod being in contact with another medium that's held at a fixed temperature.

(a) If the IBVP has the "clamped" boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0$$

prove that there are no nontrivial solutions $X(x)$ when $k \geq 0$.

(b) If the IBVP has the no-flux boundary conditions

$$u_x(0,t) = 0, \qquad u_x(L,t) = 0$$

prove that there are no nontrivial solutions $X(x)$ when $k > 0$.

(c) Consider the boundary conditions that would correspond to the bar being held in a material, at temperature T_1, at one end and in a different material, at temperature T_2, at the other end:

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ -u_x(0,t) = -\frac{h_1}{k}(u(0,t) - T_1) \\ u_x(L,t) = -\frac{h_2}{k}(u(L,t) - T_2) \\ u(x,0) = \phi(x) \end{cases}$$

i. Why do I have two different hs in the boundary conditions?

ii. Find $u_{ss}(x)$, the steady state for the IBVP. Is there always a steady state? Write $u_{ss}(0)$ as $a_{11}T_1 + a_{12}T_2$ and $u_{ss}(L)$ as $a_{21}T_1 + a_{22}T_2$. Is there a relationship between a_{11} and a_{12}? Between a_{21} and a_{22}? Is there a relationship between the pair (a_{11}, a_{12}) and the pair (a_{21}, a_{22})? If $h_1 = 0$, what is u_{ss}? Does your answer make sense? If $h_2 = 0$, what is u_{ss}? Does your answer make sense?

iii. Subtract the steady state from $u(x,t)$ to find an IBVP for $U(x,t)$:

$$\begin{cases} U_t = \alpha^2 U_{xx} \\ -U_x(0,t) = -\frac{h_1}{k}U(0,t) \\ U_x(L,t) = -\frac{h_2}{k}u(L,t) \\ u(x,0) = \phi(x) - u_{ss}(x) \end{cases}$$

We seek to solve this IBVP by separation of variables. Prove that there are no nontrivial solutions $X(x)$ when $k > 0$. What about if $k = 0$?

(d) In the previous problem set, you considered the IBVP

$$\begin{cases} u_t = \alpha^2 u_{xx} \quad \text{for } 0 < x < 2,\ 0 < t \\ u_x(0,t) + 3u(0,t) = 0 \quad \text{for } 0 < t \\ u_x(2,t) - 8u(2,t) = 0 \quad \text{for } 0 < t \\ u(x,0) = \phi(x) \end{cases}$$

There are two nontrivial $X(x)$ which correspond to $k > 0$ solutions. Find them.