Mat1062: Computational Methods for PDE

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1 Ownership

These notes are primarily those of Rob Almgren who taught an analogous course in 2003. Whatever you learn of value from them is due to him. All mistakes and sources of confusion are to be blamed on me.

2 Alternating Direction Implicit methods

Here is one approach to avoiding solving the linear system for an implicit discretization on a rectangular grid. We need to introduce some notation.

The first-order differences of a grid function $u_{i,j}$ are

$$\begin{aligned} (\delta_x^+ u)_{ij} &= u_{i+1,j} - u_{i,j} & \qquad (\delta_y^+ u)_{ij} &= u_{i,j+1} - u_{i,j} \\ (\delta_x^- u)_{ij} &= u_{i,j} - u_{i-1,j} & \qquad (\delta_y^- u)_{ij} &= u_{i-1,j} - u_{i,j}. \end{aligned}$$

The second-order differences are

$$\begin{aligned} (\delta_x^2 u)_{i,j} &= (\delta_x^+ \delta_x^- u)_{i,j} = (\delta_x^- \delta_x^+ u)_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \\ (\delta_y^2 u)_{i,j} &= (\delta_y^+ \delta_y^- u)_{i,j} = (\delta_y^- \delta_y^+ u)_{i,j} = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}. \end{aligned}$$

These are operators, so the second set literally means to apply first one then the other. They are essentially what is implemented in Matlab's `diff` operation, except for boundary effects which we are ignoring here. As noted above, these operators *commute* with each other, including mixtures of $x-$ and $y-$differences: $\delta_x^2 \delta_y^2$ is the same operation as $\delta_y^2 \delta_x^2$, etc.

Then our grid approximation to the Laplacian becomes

$$Lu = \frac{1}{h^2} \left(\delta_x^2 + \delta_y^2 \right)$$

and our fully explicit approximation may be written

$$u^{n+1} \;=\; \left(I \,+\, \lambda\delta_x^2 \,+\, \lambda\delta_y^2 \right) u^n$$

The Crank-Nicolson scheme ($\theta = \frac{1}{2}$) may be written

$$\left(I \,-\, \tfrac{1}{2}\lambda\delta_x^2 \,-\, \tfrac{1}{2}\lambda\delta_y^2 \right) u^{n+1} \;=\; \left(I \,+\, \tfrac{1}{2}\lambda\delta_x^2 \,+\, \tfrac{1}{2}\lambda\delta_y^2 \right) u^n$$

ADI methods split the operator on the left and right sides, in a way that does not change the nature of the approximation very much, but makes the system much easier to solve. The simplest such is the *Peaceman-Rachford* scheme (1955):

$$\left(I \,-\, \tfrac{1}{2}\lambda\delta_x^2 \right) \left(I - \tfrac{1}{2}\lambda\delta_y^2 \right) u^{n+1} \;=\; \left(I \,+\, \tfrac{1}{2}\lambda\delta_x^2 \right) \left(I + \tfrac{1}{2}\lambda\delta_y^2 \right) u^n.$$

This is not quite the same as the Crank-Nicolson scheme, since an extra term $\frac{1}{4}\lambda^2\delta_x^2\delta_y^2 u^{n+1}$ has been added on the left, and $\frac{1}{4}\lambda^2\delta_x^2\delta_y^2 u^n$ has been added on the right. Thus the additional contribution to the truncation error is

$$\tfrac{1}{4}\lambda^2\delta_x^2\delta_y^2\delta_t^+ u \;\sim\; \tfrac{1}{4}\lambda^2 h^4 k\, u_{txxyy} \;\sim\; \mathcal{O}(k^3)$$

in which we have used $\delta_x^2 \sim h^2\partial_{xx}$, $\delta_y^2 \sim h^2 u\partial_{yy}$, and $\delta_t^+ u = u^{n+1} - u^n \sim k u_t$. Thus the additional error introduced by this splitting is of the same size as the truncation error $\mathcal{O}(k^3)$ already present in the second-order CN scheme: the modified scheme is still second-order accurate.

The Peaceman-Rachford modification is easily solved in two stages by using an intermediate variable $u^{n+1/2}$ and writing it as

$$\begin{aligned} \left(I \,-\, \tfrac{1}{2}\lambda\delta_x^2 \right) u^{n+\frac{1}{2}} \;&=\; \left(I \,+\, \tfrac{1}{2}\lambda\delta_y^2 \right) u^n \\ \left(I \,-\, \tfrac{1}{2}\lambda\delta_y^2 \right) u^{n+1} \;&=\; \left(I \,+\, \tfrac{1}{2}\lambda\delta_x^2 \right) u^{n+\frac{1}{2}}. \end{aligned}$$

To see that this is equivalent to the above unsplit form, operate on the first equation with $I + \frac{1}{2}\lambda\delta_x^2$ and on the second with $I - \frac{1}{2}\lambda\delta_x^2$, and use commutativity. Each of these two equations is a tridiagonal system and can be solved easily, accurately, and rapidly.

This modification does not destroy the stability properties of the Crank-Nicolson scheme, as may easily be checked by looking for exact solutions $u_{i,j}^n = \omega_1^i \omega_2^j \eta^n$. We find

$$\eta \;=\; \frac{\big(1 - \lambda(1 - \mathrm{Re}\,\omega_1)\big)\big(1 - \lambda(1 - \mathrm{Re}\,\omega_2)\big)}{\big(1 + \lambda(1 - \mathrm{Re}\,\omega_1)\big)\big(1 + \lambda(1 - \mathrm{Re}\,\omega_2)\big)}$$

instead of the CN form

$$\eta_{\rm CN} \;=\; \frac{\big(1-\lambda(1-\mathrm{Re}\,\omega_1)-\lambda(1-\mathrm{Re}\,\omega_2)\big)}{\big(1+\lambda(1-\mathrm{Re}\,\omega_1)+\lambda(1-\mathrm{Re}\,\omega_2)\big)}$$

but both have $|\eta| \le 1$ for all λ.