

Mat1062: Computational Methods for PDE

Problem Set 5

Friday March 21, 2008

due: Tuesday April 1, 2008

1. We want to solve the Poisson equation $-\Delta u = f$ in the unit square $\Omega = [0, 1]^2$, with $u = 0$ on the boundaries of the square. It is easy to see that this problem can be posed in three ways just as in the optional Problem 3, with the modifications that $a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v$ only, and in (b) and (c) we add the requirement that v should vanish on the boundaries.

Choose an integer N , and define a uniform distribution of nodes

$$(x_i, y_j) = (ih, jh) \quad \text{where} \quad h = 1/N$$

There are two particularly simple ways to construct the approximating function space:

- (a) The interpolating functions $\phi_{i,j}(x, y)$ are bilinear:

$$\phi_{i,j}(x, y) = \phi_i(x) \phi_j(y)$$

where ϕ_j is the standard 1-D “hat” function

$$\phi_j(y) = \begin{cases} \frac{1}{h}(y - y_{j-1}) & y_{j-1} \leq y \leq y_j \\ -\frac{1}{h}(y - y_{j+1}) & y_j < y \leq y_{j+1} \\ 0 & \text{otherwise} \end{cases}$$

In this case a basis function $\phi_{i,j}$ has the value 1 at the node (x_i, y_j) , is positive on $(x_{i-1}, x_{i+1}) \times (y_{j-1}, y_{j+1})$, and vanishes elsewhere. Each basis function will overlap with eight other basis functions and so the matrix K will have eight nonzero entries per row.

- (b) Given a node (x_i, y_j) , it's at the centre of a diamond made by connecting (x_i, y_{j+1}) to (x_{i+1}, y_j) , connecting (x_{i+1}, y_j) to (x_i, y_{j-1}) , connecting (x_i, y_{j-1}) to (x_{i-1}, y_j) , and connecting (x_{i-1}, y_j) to (x_i, y_{j+1}) . A basis function $\phi_{i,j}$ has the value 1 at the node (x_i, y_j) , is positive on the interior of the diamond, and vanishes elsewhere. Construct $\phi_{i,j}$ as follows. Consider the points (x_i, y_{j+1}) , (x_{i+1}, y_j) , and

(x_i, y_j) . Find a linear function which is 1 at (x_i, y_j) and is zero on the line connecting (x_i, y_{j+1}) and (x_{i+1}, y_j) . This function defines $\phi_{i,j}$ in this triangle of the diamond. Construct $\phi_{i,j}$ in the remaining three triangles accordingly. Each basis function will overlap with four other basis functions and so the matrix K will have four nonzero entries per row.

In each case, derive the linear system that the $u_{i,j}$ should satisfy.

- Using google and books, find how to do Gaussian quadrature for a triangle Δ_0 with vertices at $(0, 0)$, $(1, 0)$, and $(0, 1)$. (Don't forget to cite!) There's one with four nodes and another one with six nodes. Code both of them up and show that they're working by testing against some specific examples. Now, given a triangle Δ with vertices at (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) find the change of coordinates that changes an integral over Δ into one over Δ_0 , allowing you to use Gaussian quadrature to approximate the integral there. Code this up and show that your code is working by testing it against some specific examples.

The following are problems for you to think about but not hand in.

- Using the matlab codes I provided, verify Carlos' observation about the two point boundary value problem in 1-d: if all you wanted to know was u at a particular point in $(0, 1)$ then you can do this by taking three nodes: 0, 1, and the point you're interested.
- Play around with the nonuniform mesh 1d code and find an example where you have a node distribution which is well-suited for your choice of f .
- Let Ω be a region in \mathbb{R}^n , with $\partial\Omega$ its boundary. For u, v, f defined in Ω as well as on $\partial\Omega$, and g defined on $\partial\Omega$, define the inner products

$$a(u, v) = \int_{\Omega} (\nabla u \cdot \nabla v + uv) dV, \quad (f, v) = \int_{\Omega} f v dV, \quad \langle g, v \rangle = \int_{\partial\Omega} g v dS$$

where dV is the interior volume element and dS is surface element. Show that the following problems are equivalent:

(a) (Helmholz PDE)

$$\begin{aligned} -\Delta u + u &= f && \text{in } \Omega \\ \frac{\partial u}{\partial n} &= g && \text{on } \partial\Omega. \end{aligned}$$

(b) For all v defined in Ω (not necessarily zero on $\partial\Omega$),

$$a(u, v) = (f, v) + \langle g, v \rangle.$$

(c) Minimize

$$F(v) = \frac{1}{2} a(v, v) - (f, v) - \langle g, v \rangle$$

over functions v defined in Ω (not necessarily zero on $\partial\Omega$).

“Show” means just integrate by parts and get the boundary terms right; don’t worry too much about the functional analysis aspects. For the record, in (b) and (c) the relevant space is $H^1(\Omega)$, and if you replace that by a finite-dimensional approximation you will immediately have yourself a finite-element discretization.