

Mat1062: Introductory Numerical Methods for PDE

Problem Set 5

Thursday March 19, 2008

due: Thursday April 2, 2008

1. You want to find approximate solutions of

$$\begin{cases} u_t = Du_{xx} & x \in (0, L), t > 0 \\ u(x, 0) = u_0(x) & x \in [0, L] \\ u(0, t) = 0 & t > 0 \\ u(L, t) = 0 & t > 0 \end{cases}$$

using the Finite-Element Ritz-Galerkin approximation. That is, you seek

$$u(x, t) = \sum_{j=1}^{N-1} c_j(t) \phi_j(x)$$

so that the residual $R(x, t) = u_t - Du_{xx}$ is orthogonal to each of your basis functions. This then determines a system of ODEs for the coefficients $c_j(t)$, which you can time-step.

- (a) What is the function space V that you should be considering for the above initial boundary value problem? What is the natural set of piecewise linear basis elements? To save yourself grief, assume that your nodes are equally spaced: $x_i - x_{i-1} = h = L/N$.
 - (b) For your choice of basis elements, find the system of ODEs that you need to solve (including the initial conditions, of course).
 - (c) Code your system up, based on explicit Euler time-stepping. Do problem 3b from the first homework set, but for this Ritz-Galerkin scheme. Also, compare the errors for this method to the errors that you got when doing problem 3b way back then. (I.e. how do the finite-difference errors compare to the finite-element errors?)
2. We wish to consider quadratic finite elements.

- (a) For $x_{-1} < x_0 < x_1$ we construct quadratic functions

$$\Phi_{-1} = \frac{(x - x_0)(x - x_1)}{(x_{-1} - x_0)(x_{-1} - x_1)}, \Phi_0 = \frac{(x - x_{-1})(x - x_1)}{(x_0 - x_{-1})(x_0 - x_1)}, \Phi_1 = \frac{(x - x_{-1})(x - x_0)}{(x_1 - x_{-1})(x_1 - x_0)}.$$

Note: Φ_{-1} is 1 at the left node and zero at the other two, Φ_0 is 1 at the center node and zero at the others, and Φ_1 is 1 at the right node and zero at the others.

Assume $x_{-1} = -1$, $x_0 = 0$, and $x_1 = 1$. Plot the three functions and find the six inner products:

$$a(\Phi_i, \Phi_j) = \int_{-1}^1 \Phi'_i(x) \Phi'_j(x) dx \quad i, j \in \{-1, 0, 1\}.$$

(b) Assume $x_{-1} = -h$, $x_0 = 0$, and $x_1 = h$. Find the six inner products:

$$a(\Phi_i, \Phi_j) = \int_{-h}^h \Phi'_i(x) \Phi'_j(x) dx \quad i, j \in \{-1, 0, 1\}.$$

(c) We now define our basis functions for the space

$$V = \left\{ v(x) \text{ real-valued functions on } \mathbb{R} \mid \int_0^1 v(x)^2 dx < \infty, \int_0^1 v'(x)^2 dx < \infty, v(0) = 0 \right\}$$

Assume $0 = x_0 < x_1 < x_2 < x_3 < x_4 = 1$. The four basis functions are

$$\phi_1(x) = \begin{cases} \Phi_0(x) & x \in [x_0, x_2] \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_2(x) = \begin{cases} \Phi_1(x) & x \in [x_0, x_2] \\ \Phi_{-1}(x) & x \in [x_2, x_4] \end{cases}$$

$$\phi_3(x) = \begin{cases} \Phi_0(x) & x \in [x_2, x_4] \\ 0 & \text{otherwise} \end{cases}$$

$$\phi_4(x) = \begin{cases} \Phi_1(x) & x \in [x_2, x_4] \\ 0 & \text{otherwise} \end{cases}$$

Plot the four basis functions. Now, assume that $0 = x_0 < x_1 < \dots < x_{2n} = 1$. What are the corresponding $2n$ basis functions? Assume the nodes are equally spaced ($x_i - x_{i-1} = h = 1/(2n)$) and compute the inner products

$$a(\phi_i, \phi_j) = \int_0^1 \phi'_i(x) \phi'_j(x) dx \quad i, j \in \{1, \dots, 2n\}.$$

(d) I would like you to reproduce the simulations done in the March 17 notes for a uniform mesh. To do this, you will need to calculate

$$\int_0^1 f(x) \phi_j(x) dx, \quad \forall j \in \{1, \dots, 2n\}.$$

Find these integrals for $f(x) = \cos(3\pi x)$. To do this, you may find the following formulae helpful:

$$\int_{-1}^1 \cos(\omega x + \psi) dx = \frac{\sin(\omega - \psi) + \sin(\omega + \psi)}{\omega}$$

$$\int_{-1}^1 x \cos(\omega x + \psi) dx = \frac{2 \sin(\psi) (\omega \cos(\omega) - \sin(\omega))}{\omega^2}$$

$$\int_{-1}^1 x^2 \cos(\omega x + \psi) dx = \frac{2 \cos(\psi) (\omega^2 \sin(\omega) + 2\omega \cos(\omega) - 2 \sin(\omega))}{\omega^3}$$

- (e) Now you have all the ingredients to write the code to reproduce the simulation presented in the March 17 notes for a uniform mesh. Please reproduce the table that has the L^∞ and L^2 norms as well as the ratios of the errors. If your computer's too slow, then skip the 641 node case.
3. What is the convergence rate that you observed? Bearing in mind what you found in problem 1 when you compared the finite-difference three-point stencil to the finite-element approach with piecewise linear basis functions, why is the convergence rate that you observed natural? If you were to find a related finite-difference approach, what would be the approximation you'd use for u_{xx} ? Prove that your guess has the right rate of convergence.