1. Consider the explicit upwind and Lax-Friedrichs schemes for the advection equation

\[ u_t + au_x = 0. \]

We say the scheme is stable with respect to the \( L^1 \) norm if

\[ ||u^{n+1}||_{L^1} \leq ||u^n||_{L^1} \quad \text{for all } n \in \mathbb{N} \]

where

\[ ||f||_{L^1} = h \sum_j |f_j| \]

(a) Show that if \( 0 \leq ak/h \leq 1 \) then the explicit upwind scheme is stable with respect to the \( L^1 \) norm.

(b) Show that if \( -1 \leq ak/h \leq 1 \) then the Lax-Friedrichs scheme is stable with respect to the \( L^1 \) norm.

2. Consider the initial data

\[ u_0(x) = \begin{cases} 
2 & x < -1 \\
1 & -1 < x < 1 \\
0 & 1 < x
\end{cases} \]

(a) Find the weak solution of \( u_t + (u^2/2)_x = 0 \) that results from this initial data. Sketch the characteristics and shock paths in the \( x-t \) plane. Hint: the two shocks merge into one shock at some point in space time.

(b) Compute the solution on the interval \([-2, 5]\). Sample codes to modify are provided on the webpage. Explain how you choose to discretize the initial data. Try all three flux functions: Explicit Upwind, Lax-Friedrichs, Lax-Wendroff. Given a solution, use “extract.m” to try and track the locations of the shocks and demonstrate that they’re where they should be.

3. Consider the conservation law

\[ u_t(x, y, t) + \nabla \cdot \left( \begin{array}{c} f_1(u(x, y, t)) \\
f_2(u(x, y, t)) \end{array} \right) = 0 \]

(a) If \( \Omega \subset \mathbb{R}^2 \) is a nice enough domain for the divergence theorem to hold, what is the conservation form of the equation in terms of \( \Omega \)?

(b) if \( \Omega = [x_{i-1/2}, x_{i+1/2}] \times [y_{k-1/2}, y_{k+1/2}] \) is a square with side-length \( h \) what is the conservation form of the equation?

(c) What are the quantities you would need to approximate in order to write a conservative numerical scheme for this equation?