Mat1062: Computational Methods for PDE

Problem Set 4

Tuesday, March 11, 2008

due: Thursday, March 20 in class

1. Consider the explicit upwind and Lax-Friedrichs schemes for the advection equation

$$u_t + au_x = 0.$$

We say the scheme is stable with respect to the L^1 norm if

$$||u^{n+1}||_{L^1} \leq ||u^n||_{L^1} \qquad \text{for all } n \in \mathbb{N}$$

where

$$||f||_{L^1} = h \sum_j |f_j|$$

   (a) Show that if $0 \leq ak/h \leq 1$ then the explicit upwind scheme is stable.

   (b) Show that if $-1 \leq ak/h \leq 1$ then the Lax-Friedrichs scheme is stable.

2. Compute the modified equation for the method

$$u_j^{n+1} = u_j^n - \frac{k}{2h} a(u_{j+1}^n - u_{j-1}^n)$$

and use it to explain why this method might be expected to be unstable. (You already know it is from the von Neumann analysis.)

3. In the class notes, I gave the conservative flux function F for the nonlinear analogues of: explicit upwind, Lax-Friedrichs, and Lax-Wendroff. Find F for Beam-Warming and give conditions under which you have a consistent scheme.

4. Consider the conservation law

$$u_t(x, y, t) + \nabla \cdot \begin{pmatrix} f_1(u(x, y, t)) \\ f_2(u(x, y, t)) \end{pmatrix} = 0$$

   (a) If $\Omega \subset \mathbb{R}^2$ is a nice enough domain for the divergence theorem to hold, what is the conservation form of the equation in terms of Ω?

(b) if $\Omega = [x_{i-1/2}, x_{i+1/2}] \times [y_{k-1/2}, y_{k+1/2}]$ is a square with side-length h what is the conservation form of the equation?

(c) What are the quantities you would need to approximate in order to write a numerical scheme for this equation?

5. Solve $u_t + (1/2\ u^2)_x = 0$ with initial data

$$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 \le x < 2 \\ 0 & 2 \le x \end{cases}$$

Sketch the characteristics and shock paths in the x-t plane. Hint: the two shocks merge into one shock at some point in space time.