Mat1062: Introductory Numerical Methods for PDE

Problem Set 4

Thursday, Feb 26, 2009

due: Tuesday, March 17 in class

1. For each of the following initial value problems, find the solution $u(x,t)$:

   (a)
   $$\begin{cases} u_t + xu_x = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \sin(x) & x \in \mathbb{R} \end{cases}$$

   (b)
   $$\begin{cases} u_t + tu_x = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \sin(x) & x \in \mathbb{R} \end{cases}$$

   (c)
   $$\begin{cases} u_t + tu_x = 2u & x \in \mathbb{R}, t > 0 \\ u(x,0) = \sin(x) & x \in \mathbb{R} \end{cases}$$

   (d)
   $$\begin{cases} u_t + tu_x = u^2 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \sin(x) & x \in \mathbb{R} \end{cases}$$

2. Consider the initial data
   $$u_0(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 \le x < 2 \\ 0 & 2 \le x \end{cases}$$

   (a) Find the weak solution of $u_t + (1/2\ u^2)_x = 0$ that results from this initial data. Sketch the characteristics and shock paths in the x-t plane. Hint: the two shocks merge into one shock at some point in space time.

   (b) Find the weak solution of $(u^2)_t + (2/3\ u^3)_x = 0$ that results from this initial data. Sketch the characteristics and shock paths in the x-t plane. Hint: the two shocks merge into one shock at some point in space time.

3. In the class notes, I gave the conservative flux function F for the nonlinear analogues of: explicit upwind, Lax-Friedrichs, and Lax-Wendroff. Find F for Beam-Warming and give conditions under which you have a consistent scheme.

4. Consider the conservation law

$$u_t(x,y,t) + \nabla \cdot \begin{pmatrix} f_1(u(x,y,t)) \\ f_2(u(x,y,t)) \end{pmatrix} = 0$$

   (a) If $\Omega \subset \mathbb{R}^2$ is a nice enough domain for the divergence theorem to hold, what is the conservation form of the equation in terms of Ω?

   (b) if $\Omega = [x_{i-1/2}, x_{i+1/2}] \times [y_{k-1/2}, y_{k+1/2}]$ is a square with side-length h what is the conservation form of the equation?

   (c) What are the quantities you would need to approximate in order to write a conservative numerical scheme for this equation?