

## Mat1062: Introductory Numerical Methods for PDE

### Problem Set 3

Thursday, February 12, 2008

due: Thursday, February 26 in class

1. In the first homework set, you computed the solution of the heat equation on an interval  $[-L, L]$  with initial data

$$u_0(x) = \begin{cases} \frac{1}{h} & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

You did explicit ( $\theta = 0$ ) time-stepping and considered both Dirichlet and Neumann boundary conditions. You expected to see a spreading Gaussian of mass 1, at least for a little while:

$$u(x, t) \sim \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}.$$

- (a) If your solution exactly equal to a spreading Gaussian then at each moment in time you could plot your solution in some way so that at each moment in time the profiles would look the same. *Hint: given a particular time  $t^n$  and profile  $u^n$  the first thing you do is create a new profile of height 1. You'll need to figure out some way to deal with space.* Do this for a sequence of times, showing a time or two which is “too early” for this collapse onto a Gaussian, some times which show pretty good collapses, and a time or two which is “too late” for this collapse onto a Gaussian.
  - (b) Instead of showing profiles, for each time  $t^n$  you could find the height  $H^n$  (easily done) and the half-width (less easily done). You could then plot the each of them independently of each other ( $\log_{10}(t^n)$  versus  $\log_{10}(H^n)$ , say) and you can then plot them against each other in some way to see the time range in which you have reasonable collapse onto Gaussians. Whenever possible, do things in terms of logarithms.
2. Consider the explicit upwind and Lax-Friedrichs schemes for the advection equation

$$u_t + au_x = 0.$$

We say the scheme is stable with respect to the  $L^1$  norm if

$$\|u^{n+1}\|_{L^1} \leq \|u^n\|_{L^1} \quad \text{for all } n \in \mathbb{N}$$

where

$$\|f\|_{L^1} = h \sum_j |f_j|$$

- (a) Show that if  $0 \leq ak/h \leq 1$  then the explicit upwind scheme is stable.
  - (b) Show that if  $-1 \leq ak/h \leq 1$  then the Lax-Friedrichs scheme is stable.
3. Compute the modified equation for the method

$$u_j^{n+1} = u_j^n - \frac{k}{2h} a(u_{j+1}^n - u_{j-1}^n)$$

and use it to explain why this method might be expected to be unstable. (You already know it is from the von Neumann analysis.)